Covering and gluing of algebras and differential algebras

Dirk Calow* and Rainer Matthes†

Institut für Theoretische Physik
Fakultät für Physik und Geowissenschaften
Universität Leipzig
Augustusplatz 10/11
D-04109 Leipzig
Germany

Abstract

Extending work of Budzyński and Kondracki, we investigate coverings and gluings of algebras and differential algebras. We describe in detail the gluing of two quantum discs along their classical subspace, giving a C^*-algebra isomorphic to a certain Podleś sphere, as well as the gluing of $U_{q^{1/2}}(sl_2)$-covariant differential calculi on the discs.

1991 MSC: 81R50, 46L87
Keywords: covering; gluing; C*-algebra; differential algebra; quantum disc; quantum sphere; covariance

1 Introduction

In classical geometry or topology, the objects of study can always be considered as being glued together from local pieces which are in a certain sense (e.g. topologically) trivial. The extension of this idea to noncommutative geometry seems to be not straightforward. A possible point of view suggested by the commutative situation is to describe subspaces by ideals and to glue algebras along ideals, using a pull-back (fibered product) construction. This is the starting point of Budzyński and Kondracki in [2], where coverings of C^*-algebras and locally trivial principal fibre bundles over such covered algebras have been introduced. In view of the fact that there are nontrivial algebras (e.g. the irrational rotation algebra) which have no nontrivial ideals, such an approach cannot reflect all aspects of topological nontriviality of noncommutative algebras. However, there are many algebras, in particular in the field of quantum groups and quantum spaces, which have enough ideals for performing gluing procedures, and it seems to be worthwhile to explore which kind of examples may arise in this way.

If one wants to do differential geometry in this scheme, the differential algebras (defining the differential structure) should also have a covering adapted to the covering of the underlying algebra. The construction of such adapted differential algebras will be one of the main aims of this article. It will be used in a subsequent paper, where we will consider differential structures and connections on locally trivial quantum principal fibre bundles in the sense of [2].

The present paper starts with the definition of coverings of algebras. Since we want to apply this notion also for differential algebras, we cannot restrict ourselves to C^*-algebras, which leads to a difficulty being absent there: It may happen that a natural gluing procedure fails to lead

*supported by Deutsche Forschungsgemeinschaft, e-mail Dirk.Calow@itp.uni-leipzig.de
†supported by Sächsisches Staatsministerium für Wissenschaft und Kunst,
e-mail Rainer.Matthes@itp.uni-leipzig.de

from the collection of algebras corresponding to quantum subspaces defined by a covering back to the original algebra. Coverings, which do not have this pathology, we call complete coverings. All the coverings of an algebra are complete if the operations $+$ and $\bigcap$ between ideals are distributive with respect to each other. If this general property of ideals is not assumed, we can still give slightly weakened criteria for the completeness of a covering. If an algebra is defined as a gluing, it has always a natural complete covering.

For differential algebras $\Gamma(B) = \bigoplus_{n\in\mathbb{N}} \Gamma^n(B)$ over an algebra B, we only consider differential coverings being nontrivial with respect to B, i. e. coverings consisting of differential ideals whose components in degree zero form a nontrivial covering of B. We show that for an algebra B with given covering $(J_i)_{i=1,\ldots,n}$ and for given differential calculi $\Gamma(B_i)$ over the algebras $B_i = B/J_i$ corresponding to the "local pieces" of B there exists a unique differential calculus $\Gamma(B)$ such that the natural projections $\pi_i : B \longrightarrow B_i$ are differentiable and the kernels of the differential extensions of the π_i form a covering of $\Gamma(B)$. The covering completion with respect to this covering is in general only locally a differential calculus. We also give a criterion assuring that the differential ideals generated by the ideals of a covering of an algebra B form a covering of a given differential algebra $\Gamma(B)$.

The second part of the paper is devoted to an example. Gluing together two copies of a quantum disc we obtain a C^*-algebra isomorphic to the C^*-algebra of the Podleś sphere $S^2_{\mu c}$, $c > 0$. This isomorphism, already mentioned in [2], relies on the isomorphy of the disc algebra with the C^*-algebra of the one-sided shift, and on a result of Sheu [11] about the isomorphy of the gluing of two shift algebras by means of the symbol map and the Podleś sphere. We show that this C^*-algebra may also be characterized as the C^*-closure of a "polynomial" algebra given in terms of generators and relations naturally arising from generators and relations of the disc algebras via the gluing procedure. These generators should be considered as another set of "coordinate functions" on the quantum sphere, which arise via a homeomorphism from natural coordinates on a quantum version of a top of a cone. This is suggested by considering the spectra of the generators.

Finally, we construct, according to our general procedure, a differential calculus on our "quantum top" out of two $U_{q^{1/2}}(sl_2)$-covariant differential calculi over the quantum discs. This differential calculus is also described in terms of relations between the generators and their differentials. It is again a $U_{q^{1/2}}(sl_2)$-covariant differential calculus.

In the sequel, the word "algebra" always means an associative unital algebra over $\mathbb{C}$. Ideals are always two-sided, and homomorphisms are homomorphisms of algebras.

2 Coverings and gluings

Let B be an algebra and let $(J_i)_{i\in I}$ be a finite family of ideals contained in B. Then the algebras B_i, B_{ij} and B_{ijk} are defined as the factor algebras of B with respect to the ideals J_i, $J_i + J_j$ and $J_i + J_j + J_k$. The corresponding natural projections are denoted by

$$\begin{aligned} \pi_i : B &\longrightarrow B_i \\ \pi_{ij} : B &\longrightarrow B_{ij} \\ \pi_{ijk} : B &\longrightarrow B_{ijk}. \end{aligned}$$

There are canonical surjective homomorphisms

$$\begin{aligned} \pi^i_j : B_i &\longrightarrow B_{ij} \\ \pi^i_{jk} : B_i &\longrightarrow B_{ijk} \\ \pi^{ij}_k : B_{ij} &\longrightarrow B_{ijk}. \end{aligned}$$

For example, $\pi_j^i(b+J_i) = b+J_i+J_j$. Obviously, one can construct analogous factor algebras and surjective homomorphisms for a higher number of indices. One easily shows $\pi_j^i \circ \pi_i = \pi_i^j \circ \pi_j = \pi_{ij}$ (and similar formulas). Furthermore, there are canonical isomorphisms $B_{ij} \simeq B_i/\pi_i(J_j)$, which map $b+J_i+J_j \in B_{ij}$ onto $\pi_i(b)+\pi_i(J_j) \in B_i/\pi_i(J_j)$. Note that the above definitions also mean $B_{ii} = B_i$, $\pi_i^i = \pi_i$, etc..

Definition 1 *Let $(J_i)_{i\in I}$ be a finite family of ideals of an algebra B. $(J_i)_{i\in I}$ is called covering of B if*

$$\bigcap_i J_i = \{0\}.$$

A covering is called nontrivial if $J_i \neq \{0\} \quad \forall i \in I$.

For C^*-algebras and closed ideals, this definition was given by Budzyński and Kondracki [2]. For commutative C^*-algebras, this notion of covering corresponds to coverings of the underlying topological space by closed sets, the ideals just consisting of the functions vanishing on the corresponding set. We want to use the definition also for differential algebras, which cannot be made C^*-algebras in an obvious way. Thus we are forced to stay in the general algebraic context of our definition. As a consequence there may arise difficulties with a reconstruction of the algebra from a covering by a gluing procedure which is always possible for C^*-algebras (see Proposition 6 below). This is the motivation for introducing the notion of a complete covering.

Definition 2 *Let B be an algebra and let $(J_i)_{i\in I}$ be a covering of B. The algebra*

$$B_c := \{(a_i)_{i\in I} \in \bigoplus_{i\in I} B_i | \pi_j^i(a_i) = \pi_i^j(a_j)\}. \tag{1}$$

is called the covering completion of B with respect to $(J_i)_{i\in I}$. The covering $(J_i)_{i\in I}$ is called complete if the injective homomorphism

$$K : B \longrightarrow B_c \tag{2}$$

defined by $K(a) = (\pi_i(a))_{i\in I}$ is surjective. $p_i : B_c \longrightarrow B_i$ denotes the restriction of the canonical projection $pr_i : \bigoplus_{j\in I} B_j \longrightarrow B_i$.

The name "covering completion" for B_c is justified by the fact that $(ker\, p_k)_{k\in I}$ is a complete covering of B_c, which will be a special case of Proposition 8. p_i is surjective, since π_i is surjective and $\pi_i = p_i \circ K$.

Notice that the condition defining a complete covering is very similar to one of the sheaf axioms: It just says that a set of locally given objects which coincide "on intersections" make up a global object. The other sheaf axiom, which says that global objects, which coincide locally, also coincide globally, corresponds to the injectivity of K, being true for any covering.

As shows the example below, there exist noncomplete coverings. On the other hand, if an algebra has a covering, it also has a complete one: If $(J_i)_{i\in I}$ is a nontrivial covering, consider $\mathcal{I} = \{I' \subset I | \bigcap_{i\in I'} J_i = 0\}$. Since the index set I is finite, there exists $I' \in \mathcal{I}$ with minimal cardinality, i. e. $card\, I' = min_{I''\in\mathcal{I}} card\, I'' > 1$ (since the covering is nontrivial). It follows that $\bigcap_{i\in I'} J_i = 0$, $\bigcap_{i\in I''} J_i \neq 0$, $\bigcap_{i\in I'\setminus I''} J_i \neq 0$ for any $I'' \subset I'$, $I'' \neq I'$. Thus, $(\bigcap_{i\in I''} J_i, \bigcap_{i\in I'\setminus I''} J_i)$ is a nontrivial covering. Now, every two-element covering is complete:

Proposition 1 *Let B be an algebra and J_1, J_2 ideals of B. Then the mapping $K : B \longrightarrow \{(a_1, a_2) \in B/J_1 \bigoplus B/J_2 | \pi_2^1(a_1) = \pi_1^2(a_2)\}$ given by $K(b) = (\pi_1(b), \pi_2(b))$ is surjective. In particular, every covering consisting of two ideals is complete.*

Proof: One has to show that for every pair $(a_1, a_2) \in B_1 \bigoplus B_2$ fulfilling $\pi_2^1(a_1) = \pi_1^2(a_2)$ there exists an element $a \in B$ such that $\pi_1(a) = a_1$ and $\pi_2(a) = a_2$.
First we choose $\tilde{a_1},\ \tilde{a_2} \in B$ satisfying $\pi_1(\tilde{a_1}) = a_1$ and $\pi_2(\tilde{a_2}) = a_2$. Clearly, there exist elements $r_1 \in J_1$ and $r_2 \in J_2$ such that

$$\tilde{a_1} = \tilde{a_2} + r_1 + r_2$$

and one obtains the element a by

$$a = \tilde{a_1} - r_1 = \tilde{a_2} + r_2.$$

$\square$

The following proposition, which is a direct consequence of Theorems 17 and 18 (pages 279 and 280) of [13], gives a sufficient condition for the completeness of coverings:

Proposition 2 *Assume that the operations + and $\bigcap$ in the set of ideals of B are distributive with respect to each other, i. e. the set of ideals is a distributive lattice with respect to these operations. Then every covering of B is complete.*

This proposition is also true for subsets of the set of ideals of B which are closed under + and $\cap$, with + and $\cap$ distributive on the subset.

We will use similar arguments as in [13] to prove criteria for the completeness of a covering if the above condition is not assumed.

Proposition 3 *Let B be an algebra and let $(J_i)_{i\in I}$ be a covering of B. Assume that the index set is $I = \{1, 2, ...n\}$ and that the ideals satisfy*

$$\bigcap_{i=1,2...k-1} (J_i + J_k) = (\bigcap_{i=1,2,...,k-1} J_i) + J_k, \quad \forall k \in I.$$

Then $(J_i)_{i\in I}$ is complete.

Proof: Notice that the condition $\pi_j^i(a_i) = \pi_i^j(a_j)$ just means $b_i - b_j \in J_i + J_j$ for $a_i = \pi_i(b_i)$, $a_j = \pi_j(b_j)$. Thus, in order to prove surjectivity of the map $a \longrightarrow (a + J_i)_i$, we have to show that from $b_i - b_j \in J_i + J_j$ for every pair of indices follows the existence of $b \in B$ with $b - b_i \in J_i$ (for all i). This is done inductively: Induction starts with Proposition 1. Assume now that we have found an $a \in B$ with $a - a_i \in J_i$ for $i = 1, \ldots, k$. Then we have

$$a - a_{k+1} = a - a_i + a_i - a_{k+1} \in J_i + J_i + J_{k+1} = J_i + J_{k+1},$$

thus

$$a - a_{k+1} \in \bigcap_{i=1}^{k}(J_i + J_{k+1}) = (\bigcap_{i=1}^{k} J_i) + J_{k+1}.$$

According to Proposition 1 there exists $b \in B$ with $b - a \in \bigcap_{i=1}^{k} J_i$ and $b - a_{k+1} \in J_{k+1}$. Therefore $b - a_i = b - a + a - a_i \in \bigcap_{i=1}^{k} J_i + J_i = J_i$, i. e. b is the element we were looking for. $\square$

Proposition 4 *Let B be an algebra and let $(J_i)_{i\in I}$ be a complete covering of B. Then the family of ideals $(J_i)_{i\in I}$ has the property*

$$\bigcap_{i\neq k}(J_i + J_k) = (\bigcap_{i\neq k} J_i) + J_k \quad \forall k \in I.$$

Proof: The inclusion $J_k + \bigcap_{i\neq k} J_i \subset \bigcap_{i\neq k}(J_i + J_k)$ is true for subsets of a vector space. So we have to prove $\bigcap_{i\neq k}(J_i + J_k) \subset J_k + \bigcap_{i\neq k} J_i$ for a complete covering. Completeness of the covering means that for $(a_i)_{i\in I} \in \bigoplus_{i\in I} B_i$ with $\pi^i_j(a_i) = \pi^j_i(a_j)$ there exists a unique $a \in B$ with $\pi_i(a) = a_i$. Let $I = \{1, ..., n\}$ and assume $k = n$, without loss of generality.
Let $a \in \bigcap_{i<n}(J_i + J_n)$ and denote $a_i = \pi_i(a)$, $i = 1, \dots, n$. Then we have $\pi^n_i(a_n) = \pi^n_i \circ \pi_n(a) = \pi_{in}(a) = 0$ for $i < n$, and also $\pi^i_n(a_i) = \pi^i_n \circ \pi_i(a) = \pi_{ni}(a) = 0$ for $i < n$, from which we can conclude $(0, \dots, a_n) \in B_c$ and $(a_1, \dots, a_{n-1}, 0) \in B_c$. Obviously, $(a_1, \dots, a_{n-1}, 0) \in ker\, p_n$ and $(0, \dots, 0, a_n) \in \bigcap_{i<n} kerp_i$. On the other hand, $K : B \longrightarrow B_c$ is by assumption an algebra isomorphism and one easily verifies that it maps J_i onto $ker\, p_i$. Therefore, there are $b \in J_n$, $c \in \bigcap_{i<n} J_i$ such that $K(b) = (a_1, \dots, a_{n-1}, 0)$ and $K(c) = (0, \dots, 0, a_n)$, and we have

$$a = K^{-1}(K(b) + K(c)) = b + c \in J_n + \bigcap_{i<n} J_i.$$

□

Proposition 5 *For a covering* (J_1, J_2, J_3) *consisting of three ideals the following conditions are equivalent:*
(i) (J_1, J_2, J_3) *is a complete covering.*
(ii) $(J_i + J_k)\bigcap(J_j + J_k) = J_i \bigcap J_j + J_k$ *for one permutation* i, j, k *of* $1, 2, 3$.
(iii) $(J_i + J_k)\bigcap(J_j + J_k) = J_i \bigcap J_j + J_k$ *for every permutation* i, j, k *of* $1, 2, 3$.

Proof: Is an obvious combination of the two foregoing propositions according to the scheme (i) ⇒ (iii) ⇒ (ii) ⇒ (i). The right ⇒ is possible only for a three-element covering, because only then the conditions of Proposition 3 reduce to exactly one of the conditions of Proposition 4. □

Let us also note that a covering is always complete if the ideals are coprime, i. e. if $J_i + J_j = B$, $i \neq j$ ([1]). However, in this case there is no gluing at all.

Example of a noncomplete covering:
Consider the algebra

$$B = \mathbb{C} < x, y, z > /J,$$

where J is the ideal generated by the elements xy, yx, xz, zx, yz, zy. It follows that

$$\{1, x, x^2, \dots, y, y^2, \dots, z, z^2, \dots\}$$

is a linear basis of B. Consider the ideals J_1, J_2, J_3 generated by $x-y$, $x-z$, $y-z$, respectively. Obviously,

$$J_1 = \mathbb{C}(x-y) + \mathbb{C}x^2 + \mathbb{C}x^3 + \ldots + \mathbb{C}y^2 + \mathbb{C}y^3 + \ldots,$$
$$J_2 = \mathbb{C}(x-z) + \mathbb{C}x^2 + \mathbb{C}x^3 + \ldots + \mathbb{C}z^2 + \mathbb{C}z^3 + \ldots,$$
$$J_3 = \mathbb{C}(y-z) + \mathbb{C}y^2 + \mathbb{C}y^3 + \ldots + \mathbb{C}z^2 + \mathbb{C}z^3 + \ldots.$$

Moreover,

$$\mathbb{C}(x-y) + \mathbb{C}(x-z) = \mathbb{C}(x-y) + \mathbb{C}(y-z) = \mathbb{C}(x-z) + \mathbb{C}(y-z),$$

whereas

$$\mathbb{C}(x-y) \cap \mathbb{C}(x-z) = \mathbb{C}(x-y) \cap \mathbb{C}(y-z) = \mathbb{C}(x-z) \cap \mathbb{C}(y-z) = \{0\}.$$

Therefore, we obtain

$$J_1 \cap J_2 \cap J_3 = \{0\},$$

i.e. (J_1, J_2, J_3) is a covering of B. For the sums and intersections of two of the three ideals we get

$$J_1 + J_2 = \mathbb{C}(x-y) + \mathbb{C}(x-z) + \mathbb{C}x^2 + \ldots + \mathbb{C}y^2 + \ldots + \mathbb{C}z^2 + \ldots,$$
$$J_1 + J_3 = J_2 + J_3 = J_1 + J_2,$$
$$J_1 \cap J_2 = \mathbb{C}x^2 + \mathbb{C}x^3 + \ldots,$$
$$J_1 \cap J_3 = \mathbb{C}y^2 + \mathbb{C}y^3 + \ldots,$$
$$J_2 \cap J_3 = \mathbb{C}z^2 + \mathbb{C}z^3 + \ldots.$$

We conclude that

$$(J_1 + J_3) \cap (J_2 + J_3) = J_1 + J_3,$$

whereas

$$(J_1 \cap J_2) + J_3 = \mathbb{C}(y-z) + \mathbb{C}x^2 + \ldots + \mathbb{C}y^2 + \ldots + \mathbb{C}z^2 + \ldots,$$

which is strongly contained in $J_1 + J_3$. Similarly, all other possible equalities of Propositions 3 and 4 are not satisfied, which means that the covering is not complete.

Notice that we could have introduced the additional relations $x^n = 0$, $y^n = 0$, $z^n = 0$, $n \geq 2$, for example, without changing the situation essentially. For $n = 2$ we arrive at a pure vector space situation (three subspaces such that the sum of any two contains the third). Admittedly, in this case the covering is reducible, already two of the three ideals form a covering.

Second example of a noncomplete covering:
Let $\mathbb{C}[x, y]$ be the algebra of polynomials in the (commuting) indeterminates x and y. Consider the principal ideals $J_1 = (x)$, $J_2 = (y)$, $J_3 = (x - y)$. One easily verifies that

$$J_1 + J_2 = J_1 + J_3 = (J_1 + J_2) \cap (J_1 + J_3) = \text{polynomials without constant term},$$

$$J_1 + J_2 \cap J_3 = (x) + (y(x-y)),$$

i. e. $(J_1 + J_2) \cap (J_1 + J_3) \neq J_1 + J_2 \cap J_3$. Moreover,

$$J_1 \cap J_2 \cap J_3 = (xy(x-y)).$$

Therefore, going to the factor algebra $A = \mathbb{C}[x, y]/J$, where J is the ideal generated by monomials of at least third degree, we obtain a noncomplete covering (J_1, J_2, J_3). The triple $(x-y+J_1, x-y+J_2, x+J_3)$ is an element of A_c which has no preimage in A.

Proposition 6 *Any covering of a C^*-algebra consisting of closed ideals is complete.*

Proof: The closed ideals of a C^*-algebra form a distributive lattice with respect to the operations $+$ and $\cap$, which in turn follows from the fact that in this case the product of closed ideals coincides with their intersection, see [4], 1.9.12.a.. □

The proof of the following proposition can be found in [2].

Proposition 7 *A C^*-algebra which admits a faithful irreducible representation does not admit any nontrivial covering consisting of closed ideals.*

In particular, the algebra $B(H)$ of bounded operators on a Hilbert space H does not admit a nontrivial covering. The same is obviously true for any simple algebra.

A general method to construct algebras possessing a complete covering is given by a gluing procedure:

Definition 3 *Assume that there are given finite families of algebras $(B_i)_{i\in I}$ and $(B_{ij})_{i,j\in I}$ and surjective homomorphisms $\pi^i_j : B_i \longrightarrow B_{ij}$, where $B_{ij} = B_{ji}$, $B_{ii} = B_i$ and $\pi^i_i = id$. Then the algebra*

$$\oplus_{\pi^i_j} B_i := \{(a_i)_{i\in I} \in \bigoplus_{i\in I} B_i |\ \pi^i_j(a_i) = \pi^j_i(a_j)\}$$

is called gluing of the algebras B_i with respect to the π^i_j.

For $I = \{1, 2\}$, this is known as a pull-back or a fibered product of B_1 and B_2. The covering completion B_c of an algebra B with covering $(J_i)_{i\in I}$ is just the gluing of the B/J_i with respect to the natural maps π^i_j. We will show that $(ker\, p_i)_{i\in I}$, where $p_i : \oplus_{\pi^i_j} B_i \longrightarrow B_i$ are the restrictions of the canonical projections, is a complete covering of B. However, the p_i are in general not surjective. In the classical situation, this would mean that the sets, which are glued together, are not embedded in the global object.

Lemma 1 *Let $B = \oplus_{\pi^i_j} B_i$ and $A = \oplus_{\eta^i_j} A_i$ be gluings as in Definition 3. Moreover, let $\phi_i : A_i \longrightarrow B_i$ be algebra homomorphisms.*
Assume that there exist algebra homomorphisms $\phi_{ij} = \phi_{ji} : A_{ij} \longrightarrow B_{ij}$ such that

$$\phi_{ij} \circ \eta^i_j = \pi^i_j \circ \phi_i,\ i \neq j. \tag{3}$$

Then we have $(\oplus_i\phi_i)(A) \subset B$.

Proof:
$(\oplus_i\phi_i)(A) \subset B$ means $\eta^i_j(a_i) = \eta^j_i(a_j) \ \Rightarrow\ \pi^i_j(\phi_i(a_i)) = \pi^j_i(\phi_j(a_j))$. Assuming the existence of ϕ_{ij} with (3) it follows from $\eta^i_j(a_i) = \eta^j_i(a_j)$ that $\pi^i_j \circ \phi_i(a_i) = \phi_{ij} \circ \eta^i_j(a_i) = \phi_{ij} \circ \eta^j_i(a_j) = \pi^j_i \circ \phi_j(a_j)$.
□

Notice that the ϕ_{ij} fulfilling (3) exist if and only if $ker(\eta^i_j) \subset ker(\pi^i_j \circ \phi_i),\ i \neq j$. Moreover, for the lemma it is not necessary that π^i_j and η^i_j are surjective.

Proposition 8 *Let $B = \oplus_{\pi^i_j} B_i$.*
Then $(ker\, p_i)_{i\in I}$ is a complete covering of B.

Proof: It is obvious that $(ker\, p_i)_{i\in I}$ is a covering of B. The covering completion B_c with respect to this covering is defined as

$$B_c = \{(b_i)_{i\in I} \in \bigoplus_i B/ker\, p_i |\ \eta^i_j(b_i) = \eta^j_i(b_j)\},$$

where $\eta^i_j : B/ker\, p_i \longrightarrow B/(ker\, p_i + ker\, p_j)$ are the canonical maps $(b_k)_{k\in I} + ker\, p_i \longrightarrow (b_k)_{k\in I} + ker\, p_i + ker\, p_j$.

Let $\phi_i : B/ker\, p_i \longrightarrow B_i$ be defined as $(b_k)_{k\in I} + ker\, p_i \longrightarrow b_i$. Obviously, the ϕ_i are well defined and injective.
It is now sufficient to show that $(\oplus_i\phi_i)(B_c) \subset B$ and $K \circ (\oplus_i\phi_i) = id_{B_c}$, where $K : B \longrightarrow B_c$ is the canonical embedding $(b_j)_{j\in I} \longrightarrow ((b_j)_{j\in I} + ker\, p_i)_{i\in I}$. The latter is a trivial verification. In order to verify the first claim, define $\phi_{ij} : B/(ker\, p_i + ker\, p_j) \longrightarrow B_{ij}$ by $\phi_{ij} \circ \eta^i_j = \pi^i_j \circ \phi_i$, i. e. $\phi_{ij}((b_k)_{k\in I} + ker\, p_i + ker\, p_j) := \pi^i_j(b_i)$. According to the remark after Lemma 1, ϕ_{ij} is well defined: $ker(\eta^i_j) = \{(b_k)_{k\in I} + ker\, p_i \in B/ker\, p_i |\ b_j = 0\} \subset ker\pi^i_j \circ \phi_i = \{(b_k)_{k\in I} + ker\, p_i \in B/ker\, p_i |\ b_i \in ker\pi^i_j\}$, since from $b_j = 0$ follows $\pi^i_j(b_i) = \pi^j_i(b_j) = 0$. Thus, Lemma 1 proves the claim. □

Notice that the implication $b_i = 0 \Rightarrow \pi_i^j(b_j) = 0$ also means $p_j(ker\, p_i) \subset ker(\pi_i^j)$.

Possible nonsurjectivity of p_i is reflected in nonsurjectivity of the ϕ_i appearing in the foregoing proof. In the classical situation of algebras of functions over compact spaces, this would mean that the space underlying the algebra B_i is not injectively mapped onto the space underlying $B/ker\, p_i$. With other words, the ϕ_i would encode a gluing of B_i with itself. The gluing of the $B/ker\, p_i$ does not lead to a further self-gluing, in contrast to the gluing of the B_i, where all the gluing is done "in one step".

Proposition 8 also has the consequence that the covering completion B_c of an algebra B with covering $(J_i)_{i \in I}$ has the complete covering $(ker\, p_i)_{i \in I}$.

If B_i and B_{ij} are C^*-algebras, the kernels of the homomorphisms π_j^i are closed ideals. The distributivity of $+$ and $\bigcap$ on the set of closed ideals leads to the following sufficient condition for the surjectivity of the projections p_i:

Proposition 9 *Let the algebras B_i and B_{ij} be C^*-algebras. Assume that the homomorphisms π_j^i have the following properties:*

1.

$$\pi_j^i(ker\pi_k^i) = \pi_i^j(ker\pi_k^j); \quad \forall i,j,k \in I \tag{4}$$

2. The isomorphisms $\pi_k^{ij} : B_i/(ker\pi_j^i + ker\pi_k^i) \longrightarrow B_{ij}/\pi_j^i(ker\pi_k^i)$ defined by

$$\pi_k^{ij}(f + \ker \pi_j^i + ker\pi_k^i) = \pi_j^i(f) + \pi_j^i(ker\pi_k^i)$$

fulfill

$$\pi_j^{ik^{-1}} \circ \pi_j^{ki} = \pi_k^{ij^{-1}} \circ \pi_k^{ji} \circ \pi_i^{jk^{-1}} \circ \pi_i^{kj}. \tag{5}$$

Then the homomorphisms p_i are surjective.

Remark 1: In the classical situation, i. e. $B_i = C(U_i)$, $B_{ij} = C(U_{ij})$, U_i, U_{ij} compact spaces, π_j^i is the pull-back of an embedding $\iota_j^i : U_{ij} \longrightarrow U_i$, and $ker\pi_j^i$ are the functions vanishing on $\iota_j^i(U_{ij}) \subset U_i$. Condition (4) says that $\iota_j^i(U_{ij}) \bigcap \iota_k^i(U_{ik})$ is homeomorphic to $\iota_i^j(U_{ij}) \bigcap \iota_k^j(U_{jk})$ for every triple i,j,k. π_k^{ij} is the pull-back of the restriction of ι_j^i to ${\iota_j^i}^{-1}(\iota_k^i(U_{ik}))$. (5) is the natural compatibility condition for these restrictions.

Remark 2: If B is an algebra with covering $(J_i)_{i \in I}$, then the homomorphisms $\pi_j^i : B_i = B/J_i \longrightarrow B_{ij} = B/(J_i + J_j)$ defining the covering completion B_c satisfy the assumptions of the proposition.

Proof: Assume that homomorphisms π_j^i satisfying (4) and (5) are given. Let the isomorphisms $\phi_{ij}^k : B_j/(ker\pi_i^j + ker\pi_k^j) \longrightarrow B_i/(ker\pi_j^i + ker\pi_k^i)$ be defined by $\phi_{ij}^k := \pi_k^{ij^{-1}} \circ \pi_k^{ji}$. The ϕ_{ij}^k satisfy (see formula (5)) $\phi_{ik}^j = \phi_{ij}^k \circ \phi_{jk}^i$ and ${\phi_{ij}^k}^{-1} = \phi_{ji}^k$. To prove the surjectivity of the projection p_i one has to show that for all $f \in B_i$ there exists a family $(f_k)_{k \in I} \in B$ such that $p_i((f_k)_{k \in I}) = f$. Suppose that the index set is $I = \{1, 2, 3, ..., n\}$. It is sufficient to consider the case $i = 1$. For $f \in B_1$ there exist $f_2 \in B_2$ such that $\pi_2^1(f) = \pi_1^2(f_2)$ and $f_3 \in B_3$ such that $\pi_3^1(f) = \pi_1^3(f_3)$. It follows that

$$\begin{aligned} f + ker\pi_2^1 + ker\pi_3^1 &= \phi_{12}^3(f_2 + ker\pi_1^2 + ker\pi_3^2) \\ &= \phi_{13}^2(f_3 + ker\pi_1^3 + ker\pi_2^3) \end{aligned}$$

and, with $\phi^1_{32} = {\phi^2_{13}}^{-1} \circ \phi^3_{12}$,

$$f_3 + ker\pi^3_1 + ker\pi^3_2 = \phi^1_{32}(f_2 + ker\pi^2_1 + ker\pi^2_3),$$

thus $\pi^3_2(f_3) - \pi^2_3(f_2) = r_{23} \in \pi^3_2(ker\pi^3_1)$. Choose $\tilde{r}_{23} \in ker\pi^3_1$ such that $\pi^3_2(\tilde{r}_{23}) = r_{23}$. Then $f^1_3 := f_3 - \tilde{r}_{23}$ satisfies $\pi^1_3(f) = \pi^3_1(f^1_3)$ and $\pi^2_3(f_2) = \pi^3_2(f^1_3)$.
Now assume that a family $(f_j)_{j=1,...,k}$, $f_j \in B_j$ with $f_1 = f$ and $\pi^i_j(f_i) = \pi^j_i(f_j)$, $\forall i,j = 1,...,k$ has been found. Then there exists $f^1_{k+1} \in B_{k+1}$ with $\pi^1_{k+1}(f) = \pi^{k+1}_1(f^1_{k+1})$. Assume now that for fixed $i \in \{1,...,k-1\}$ there is given $f^i_{k+1} \in B_{k+1}$ which satisfies $\pi^j_{k+1}(f_j) = \pi^{k+1}_j(f^i_{k+1})$ for $j = 1,...,i$. It follows that there exists $f^{i+1}_{k+1} \in B_{k+1}$ which satisfies $\pi^j_{k+1}(f_j) = \pi^{k+1}_j(f^{i+1}_{k+1})$, $\forall j = 1,...,i+1$: There are the identities

$$\begin{aligned} f_j + ker\pi^j_{k+1} + ker\pi^j_{i+1} &= \phi^{i+1}_{j,k+1}(f^i_{k+1} + ker\pi^{k+1}_j + ker\pi^{k+1}_{i+1}),\ \forall j = 1,...,i, \\ &= \phi^{k+1}_{j,i+1}(f_{i+1} + ker\pi^{i+1}_j + ker\pi^{i+1}_{k+1}),\ \ \forall j = 1,...,i, \end{aligned}$$

which lead to

$$f^i_{k+1} + ker\pi^{k+1}_j + ker\pi^{k+1}_{i+1} = \phi^j_{k+1,i+1}(f_{i+1} + ker\pi^{i+1}_j + ker\pi^{i+1}_{k+1}),\ \ \forall j = 1,...,i,$$

and it follows that

$$\pi^{k+1}_{i+1}(f^i_{k+1}) - \pi^{i+1}_{k+1}(f_{i+1}) = r_{i+1,k+1} \in \bigcap_{j=1,...,i} \pi^{k+1}_{i+1}(ker\pi^{k+1}_j).$$

Because of

$$\bigcap_{j=1,...,i} (ker\pi^{k+1}_j + ker\pi^{k+1}_{i+1}) = (\bigcap_{j=1,...,i} ker\pi^{k+1}_j) + ker\pi^{k+1}_{i+1},$$

in the case of C^*- algebras, applying π^{k+1}_{i+1} one obtains $\bigcap_{j=1,...,i} \pi^{k+1}_{i+1}(ker\pi^{k+1}_j)$
$= \pi^{k+1}_{i+1}(\bigcap_{j=1,...,i} ker\pi^{k+1}_j)$. Thus one finds $\tilde{r}_{i+1,k+1} \in \bigcap_{j=1,...,i} ker\pi^{k+1}_j$, such that $\pi^{k+1}_{i+1}(\tilde{r}_{i+1,k+1})$
$= r_{i+1,k+1}$, and $f^{i+1}_{k+1} = f^i_{k+1} - \tilde{r}_{i+1,k+1}$ satisfies $\pi^{k+1}_j(f^{i+1}_{k+1}) = \pi^j_{k+1}(f_j)$, $\forall j = 1,...,i+1$.
This means that there exists $f_{k+1} \in B_{k+1}$ satisfying

$$\pi^{k+1}_j(f_{k+1}) = \pi^j_{k+1}(f_j),\ \ \forall j = 1,...,k.$$

Continuing this procedure one obtains a family $(f_i)_{i\in I} \in B$ with
$p_1((f_i)_{i\in I}) = f$. Thus p_1 is surjective. □

If only two algebras are glued together the projections p_1 and p_2 are always surjective.

3 Adapted differential structures on algebras with covering

Definition 4 *A differential algebra $\Gamma(B)$ over an algebra B is an $\mathbb{N}-$graded algebra, i.e.*

$$\begin{aligned} \Gamma(B) &= \bigoplus_{n\in\mathbb{N}} \Gamma^n(B) \\ \Gamma^n(B)\Gamma^m(B) &\subset \Gamma^{m+n}(B), \end{aligned}$$

with

$$\Gamma^0(B) = B,$$

which is equipped with a differential, i.e. a linear map d of $\Gamma(B)$ fulfilling

$$\begin{array}{rcl} d(\Gamma^n(B)) & \subset & \Gamma^{n+1}(B) \\ d(\rho\eta) & = & (d\rho)\eta + (-1)^n \rho d\eta, \quad \rho \in \Gamma^n(B), \ \eta \in \Gamma(B) \\ d^2 & = & 0. \end{array}$$

A differential ideal $J \subset \Gamma(B)$ is an ideal of the algebra $\Gamma(B)$ such that

$$pr_i(J) \subset J \tag{6}$$

$$dJ \subset J, \tag{7}$$

where $pr_i : \Gamma(B) \longrightarrow \Gamma^i(B)$ is the canonical projection.
A homomorphism $\phi : \Gamma(B) \longrightarrow \Gamma(A)$ of differential algebras (of degree 0) is an algebra homomorphism with $\phi \circ d = d \circ \phi$ and $pr_i \circ \phi = \phi \circ pr_i$.

Differential ideals are in bijective correspondence with kernels of surjective homomorphisms of differential algebras.

Definition 5 *A differential algebra $\Gamma(B)$ over an algebra B is called differential calculus, if every element $\rho \in \Gamma^n(B)$ has the general form*

$$\rho = \sum_k a_0^k da_1^k ... da_n^k, \quad a_i^k \in B, \tag{8}$$

i. e. if B and dB generate $\Gamma(B)$ as an algebra.

If $\Gamma(B)$ is a differential algebra and $J \subset \Gamma(B)$ is a differential ideal, $\Gamma(B)/J$ is a differential algebra over $B/pr_0(J)$. If $\Gamma(B)$ is a differential calculus, $\Gamma(B)/J$ is a differential calculus over $B/pr_0(J)$.

For any algebra there exists the universal differential calculus $\Omega(B)$ over B. As is well known, every differential calculus $\Gamma(B)$ over B corresponds to a unique differential ideal $J(B) \subset \Omega(B)$ such that $\Gamma(B) \simeq \Omega(B)/J(B)$.

For the next definition see also [8].

Definition 6 *Let $\Psi : A \longrightarrow B$ be a homomorphism of algebras, and let $\Gamma(A)$, $\Gamma(B)$ be differential algebras over A, B. Ψ is called differentiable with respect to $\Gamma(A)$ and $\Gamma(B)$ if there exists a homomorphism of graded algebras $\Psi_\Gamma : \Gamma(A) \longrightarrow \Gamma(B)$ such that*

$$\Psi_\Gamma \circ d = d \circ \Psi_\Gamma,$$

$$\Psi_\Gamma|_A = \Psi,$$

i. e. if there exists an extension of Ψ to a homomorphism of differential algebras. Ψ_Γ is called extension of Ψ with respect to $\Gamma(A)$ and $\Gamma(B)$.

The following statements are well known or obvious:
If $\Gamma(A)$ is a differential calculus, the extension Ψ_Γ is uniquely determined by

$$\Psi_\Gamma(a_0 da_1 ... da_n) = \Psi(a_0) d\Psi(a_1) ... d\Psi(a_n). \tag{9}$$

If $\Gamma(A) = \Omega(A)$, the universal differential calculus over A, Ψ_Γ always exists (see [10]). We denote it sometimes by $\Psi_{\Omega\to\Gamma}$. If Ψ is surjective, $\Psi_{\Omega\to\Gamma}$ is surjective if and only if $\Gamma(B)$ is a differential calculus. The extension of Ψ with respect to the universal differential calculi $\Omega(A)$ and $\Omega(B)$ will be denoted by Ψ_Ω. If both $\Gamma(A)$ and $\Gamma(B)$ are differential calculi, Ψ is differentiable with respect to $\Gamma(A) \simeq \Omega(A)/J(A)$ and $\Gamma(B) \simeq \Omega(B)/J(B)$ if and only if $\Psi_\Omega(J(A)) \subset J(B)$.

Definition 7 *Let $\Gamma(B)$ be a differential algebra. A covering $(J_i)_{i\in I}$ of $\Gamma(B)$ is called differentiable if the J_i are differential ideals.*
A differentiable covering is complete if it is complete as a covering.

Proposition 10 *Let $(J_i)_{i\in I}$ be a differentiable covering of the differential algebra $\Gamma(B)$. Then $(pr_0(J_i))_{i\in I}$ is a covering of B.*

Proof:

$$\bigcap_i pr_0(J_i) = \{a \in B | a \in \bigcap_i J_i\} = \{0\}.$$

□

Definition 8 *If the covering of B induced by a differentiable covering of $\Gamma(B)$ is nontrivial the differential covering is said to be nontrivial with respect to B.*

One easily finds nontrivial differential coverings of the algebra of usual differential forms on a manifold, which are trivial with respect to the algebra of smooth functions on the manifold, e. g. just putting the zeroth degree to 0. The above definition is used to avoid such cases.

Definition 9 *A differential algebra $\Gamma(B)$ with a complete differentiable covering $(J_i)_{i\in I}$ which is nontrivial with respect to B is called LC differential algebra, if the factor algebras $\Gamma(B)/J_i$ are differential calculi over the algebras $B/pr_0(J_i)$.*

LC differential algebras will naturally arise from differential structures on locally trivial quantum principal fibre bundles (see [3]).

Definition 10 *Let $(B, (J_i)_{i\in I})$ be an algebra with covering, let $B_i = B/J_i$, let $\pi_i : B \longrightarrow B_i$ be the natural surjections, and let $\Gamma(B)$ and $\Gamma(B_i)$ be differential calculi such that π_i are differentiable and $(ker\pi_{i\Gamma})_{i\in I}$ is a covering of $\Gamma(B)$. Then $(\Gamma(B), (\Gamma(B_i))_{i\in I})$ is called adapted to $(B, (J_i)_{i\in I})$.*

If $(J_i)_{i\in I}$ is a nontrivial covering in this situation, $(ker\,\pi_{i\Gamma})_{i\in I}$ is nontrivial with respect to B, since $pr_0(ker\,\pi_{i\Gamma}) = J_i$.

In the classical case, where the π_i are the pull-backs of embeddings of closed submanifolds M_i into a manifold M and the Γ's are usual differential forms, the π_i are obviously differentiable, the $\pi_{i\Gamma}$ being the pull-backs on forms, and $ker\pi_{i\Gamma}$ are the differential forms vanishing on M_i, thus obviously forming a covering.

Proposition 11 *Let $(B, (J_i)_{i\in I})$ be an algebra with covering, and let $\Gamma(B_i)$ be differential calculi over the algebras B_i. Up to isomorphy there exists a unique differential calculus $\Gamma(B)$ such that $(\Gamma(B), (\Gamma(B_i))_{i\in I}))$ is adapted to $(B, (J_i)_{i\in I})$.*

Proof: As noted above there exist the extensions $\pi_{i_{\Omega\to\Gamma}} : \Omega(B) \longrightarrow \Gamma(B_i)$ of π_i defined by

$$\pi_{i_{\Omega\to\Gamma}}(a_0 da_1 ... da_n) := \pi_i(a_0) d\pi_i(a_1) ... d\pi_i(a_n).$$

Then $J(B) := \bigcap_i ker\pi_{i_{\Omega\to\Gamma}}$ is a differential ideal in $\Omega(B)$. Because of $pr_0(J(B)) = J(B) \bigcap B = 0$, $\Gamma(B) := \Omega(B)/J(B)$ is a differential calculus over B. The extensions π_{i_Γ} of π_i exist and the pair $(\Gamma(B), (\Gamma(B_i))_{i\in I})$ is adapted to $(B, (J_i)_{i\in I})$.
Let $\tilde{\Gamma}(B) = \Omega(B)/\tilde{J}$ be another differential calculus such that $(\tilde{\Gamma}(B), (\Gamma(B_i))_{i\in I})$ is adapted to $(B, (J_i)_{i\in I})$. Let $\pi_{\Omega,\tilde{\Gamma}} : \Omega(B) \longrightarrow \tilde{\Gamma}(B)$ denote the canonical quotient map. Then differentiability of π_i with respect to $\tilde{\Gamma}(B)$ and $\Gamma(B_i)$ means that there exist $\pi_{i_{\tilde{\Gamma}}} : \tilde{\Gamma}(B) \longrightarrow \Gamma(B_i)$ such that

$\pi_{i_{\tilde{\Gamma}}} \circ \pi_{\Omega,\tilde{\Gamma}} = \pi_{i_{\Omega\to\Gamma}}$. Therefore we have $\tilde{J} = ker\pi_{\Omega,\tilde{\Gamma}} \subset ker\pi_{i_{\Omega\to\Gamma}}$ $\forall i$, i. e. $\tilde{J} \subset \bigcap_i ker\pi_{i_{\Omega\to\Gamma}} = J$. On the other hand, if $\gamma \in J \setminus \tilde{J}$, then $\gamma + \tilde{J}$ is a nonzero element of $\bigcap_i ker\pi_{i_{\tilde{\Gamma}}}$, since $\pi_{i_{\tilde{\Gamma}}}(\gamma + \tilde{J}) = \pi_{i_{\Omega\to\Gamma}}(\gamma) = 0$ $\forall i$. $\square$

For a given differential calculus $\Gamma(B) = \Omega(B)/J(B)$, the induced differential calculi $\Gamma(B_i) = \Omega(B_i)/\pi_{i_\Omega}(J(B))$ in general do not form a pair $(\Gamma(B), (\Gamma(B_i))_{i\in I})$ which is adapted to $(B, (J_i)_{i\in I})$:

Proposition 12 *Let $\Gamma(B) := \Omega(B)/J(B)$ be a differential calculus over the algebra B, and let $(J_i)_{i\in I}$ be a covering of B.*
The pair $(\Gamma(B), (\Omega(B_i)/\pi_{i_\Omega}(J(B))))_{i\in I})$ is adapted to $(B, (J_i)_{i\in I})$ if and only if the differential ideal $J(B)$ has the property

$$J(B) = \bigcap_{i\in I}(J(B) + ker\pi_{i_\Omega}). \tag{10}$$

Remark 1: Obviously, not every differential ideal has this property. For example, in the case of universal differential calculi, where $J(B) = 0$, the differential ideal $\bigcap_i ker\pi_{i_\Omega}$ contains elements of $J_{\sigma(1)}J_{\sigma(2)}...J_{\sigma(n-1)}dJ_{\sigma(n)}$ in the first degree, where σ is any permutation, thus condition (10) is not satisfied.

Remark 2: $\Gamma(B_i) = \Omega(B_i)/\pi_{i_\Omega}(J(B))$ is canonically isomorphic to $\Gamma(B)/(J_i, dJ_i)$, where (J_i, dJ_i) is the differential ideal generated by J_i in $\Gamma(B)$. $((J_i, dJ_i))_{i\in I}$ is a covering of $\Gamma(B)$ iff (10) is satisfied.

Proof: First consider a differential ideal $J(B)$ fulfilling condition (10), and differential calculi $\Gamma(B_i) := \Omega(B_i)/\pi_{i_\Omega}(J(B))$. One has to prove $\bigcap_{i\in I} ker\pi_{i_{\Omega\to\Gamma}} = J(B)$.
Since $ker\pi_{i_{\Omega\to\Gamma}} = \pi_{i_\Omega}^{-1}(\pi_{i_\Omega}(J(B))) = J(B) + ker\pi_{i_\Omega}$, and since $J(B)$ fulfills (10), one direction of the assertion is proved.

Now let $(\Gamma(B), (\Gamma(B_i))_{i\in I})$ be adapted, i.e. $J(B) = \bigcap_{i\in I} ker\pi_{i_{\Omega\to\Gamma}}$. $J(B) \subset \bigcap_{i\in I}(J(B) + ker\,\pi_{i_\Omega})$ is trivially true. Conversely, $\bigcap_{k\in I} ker\pi_{k_{\Omega\to\Gamma}} \subset ker\pi_{i_{\Omega\to\Gamma}}$ and $ker\pi_{i_\Omega} \subset ker\pi_{i_{\Omega\to\Gamma}}$, and it follows that

$$\bigcap_{i\in I}((\bigcap_{k\in I} ker\pi_{k_{\Omega\to\Gamma}}) + ker\pi_{i_\Omega}) \subset \bigcap_{i\in I} ker\pi_{i_{\Omega\to\Gamma}}.$$

Thus, $J(B)$ satisfies (10). $\square$

Proposition 13 *Let $(\Gamma(B), (\Gamma(B_i))_{i\in I})$ be adapted to $(B, (J_i)_{i\in I})$. Then the covering completion of $(\Gamma(B), (ker\pi_{i_\Gamma})_{i\in I})$ is an LC differential algebra over B_c.*

Proof: Let

$$\pi^i_{j_\Gamma} : \Gamma(B)/ker\pi_{i_\Gamma} \longrightarrow \Gamma(B)/(ker\pi_{i_\Gamma} + ker\pi_{j_\Gamma})$$

be the quotient maps. Since the ideals $ker\pi_{i_\Gamma}$ are differential ideals, the factor algebras $\Gamma(B)/ker\pi_{i_\Gamma}$ and $\Gamma(B)/(ker\pi_{i_\Gamma} + ker\pi_{j_\Gamma})$ are differential calculi over B_i and B_{ij}, and the projections $\pi^i_{j_\Gamma}$ are the extensions of the projections π^i_j.

According to Definition 2 the covering completion $(\Gamma_c(B_c), (ker\pi_{i_{\Gamma_c}})_{i\in I})$ of the pair $(\Gamma(B), (ker\pi_{i_\Gamma})_{i\in I})$ has the entries

$$\Gamma_c(B_c) : \;=\; \{(\gamma_i)_{i\in I} \in \bigoplus_i \Gamma(B)/ker\pi_{i_\Gamma} | \pi^i_{j_\Gamma}(\gamma_i) = \pi^j_{i_\Gamma}(\gamma_j)\} \tag{11}$$

$$ker\pi_{k_{\Gamma_c}} : \;=\; \{(\gamma_i)_{i\in I} \in \Gamma_c(B_c) | \gamma_k = 0\}. \tag{12}$$

$\Gamma_c(B_c)$ has a natural grading coming from the grading of the differential calculi $\Gamma(B)/ker\pi_{i_\Gamma}$ and a natural differential d_c given by

$$d_c((\gamma_i)_{i\in I}) = (d\gamma_i)_{i\in I} \ \ \forall (\gamma_i)_{i\in I} \in \Gamma_c(B_c). \tag{13}$$

From $\pi_{k_\Gamma} = \pi_{k_{\Gamma_c}} \circ K_\Gamma$ (where K_Γ is the extension of $K : B \longrightarrow B_c$) and formula (13) it follows that $\pi_{k_{\Gamma_c}}$ is surjective and differentiable, thus $\Gamma_c(B_c)/ker\pi_{i_{\Gamma_c}}$ is a differential calculus isomorphic to $\Gamma(B)/ker\pi_{i_\Gamma}$. Because of $(\Gamma(B)/ker\pi_{i_\Gamma})^0 = B_i$, $\Gamma_c^0(B_c) = B_c$. $\square$

If the covering $(ker\pi_{i_\Gamma})_{i\in I}$ is complete, $\Gamma_c(B_c)$ is isomorphic to $\Gamma(B)$ as differential calculus.

Since the differential calculi $\Gamma(B_i)$ and $\Gamma(B)/(ker\pi_{i_\Gamma})$ are canonically isomorphic, formula (11) is the same as

$$\Gamma_c(B) = \{(\gamma_i)_{i\in I} \in \bigoplus_i \Gamma(B_i) | \pi^i_{j_\Gamma}(\gamma_i) = \pi^j_{i_\Gamma}(\gamma_i)\}.$$

We also note that the differential ideal $J(B_{ij}) \subset \Omega(B_{ij})$ corresponding to $\Gamma(B_{ij}) := \Gamma(B)/(ker\pi_{i_\Gamma} + ker\pi_{j_\Gamma})$ is

$$\begin{aligned} J(B_{ij}) &= \pi_{ij_\Omega}(ker\pi_{i_{\Omega\to\Gamma}} + ker\pi_{j_{\Omega\to\Gamma}}) \\ &= \pi^i_{j_\Omega} \circ \pi_{i_\Omega}(ker\pi_{i_{\Omega\to\Gamma}}) + \pi^j_{i_\Omega} \circ \pi_{j_\Omega}(ker\pi_{j_{\Omega\to\Gamma}}), \end{aligned}$$

and because of $\pi_{i_\Omega}(ker\pi_{i_{\Omega\to\Gamma}}) = J(B_i)$ one can also write

$$J(B_{ij}) = \pi^i_{j_\Omega}(J(B_i)) + \pi^j_{i_\Omega}(J(B_j)). \tag{14}$$

All the above considerations remain unchanged if one considers $*$-algebras, $*$-ideals, $*$-homomorphisms and differentials commuting (or anticommuting) with $*$ ($*$-differential algebras).

4 Example

Here we present an example of a quantum space being glued together from two copies of a quantum disc. The result is a C^*-algebra isomorphic to the algebra of the Podleś spheres $S^2_{\mu c}$, $c > 0$. An analogous construction was performed in [2], using another kind of quantum disc, and it was mentioned there in a footnote, that the resulting C^*-algebra is isomorphic to a Podleś sphere. To prove this isomorphy, we start with a result of Sheu [11], saying that the Podleś spheres $S^2_{\mu c}$, $c > 0$ are isomorphic as C^*-algebras to the fibered product $C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S})$ of two shift algebras by means of the symbol map σ. In our terminology, two copies of $C^*(\mathfrak{S})$ are glued together using the homomorphism $\sigma : C^*(\mathfrak{S}) \longrightarrow C(S^1)$. On the other hand, using results and arguments from [5] and [6], it is easy to show that quantum discs are as C^*-algebras isomorphic to shift algebras, and that the symbol map is transported into a natural homomorphism of the quantum disc onto $C(S^1)$, which just corresponds to the classical circle contained in the quantum disc. This gives the desired isomorphy. Moreover, using the generators of the quantum disc, we get a description of the glued C^*-algebra in terms of generators and relations. We argue that these generators should be considered as natural "coordinates" on a quantum version of a top of a cone. Finally, we apply our prescription of gluing together differential calculi. Starting from $U_{q^{1/2}}(sl_2)$-covariant differential calculi on the discs, we obtain a differential calculus on our glued quantum top, which can also be characterized in terms of generators and relations, and is also $U_{q^{1/2}}(sl_2)$-covariant.

Definition 11 *The C^*-algebra $C(D_q)$, $0 < q < 1$, of the quantum disc D_q is defined as the C^*-closure of the algebra $P(D_q) := \mathbb{C} < x, x^* > /J_q$, where J_q is the two-sided ideal in the free algebra $\mathbb{C} < x, x^* >$ generated by the relation*

$$x^* x - q x x^* = (1-q)1. \tag{15}$$

This is a one-parameter subfamily of the two-parameter family of quantum discs described in [6]. The C*-closure is formed using only bounded *-representations of $P(D_q)$. This is possible, because $\|\rho(x)\| = 1$ for every bounded *-representation, as is shown in [6]. From there, we also have

Proposition 14 *Every irreducible *-representation of $C(D_q)$ is unitarily equivalent to one of the following representations:*
(i) a one-dimensional representation ρ_θ, defined by $\rho_\theta(x) = e^{i\theta}$, $\rho_\theta(x^*) = e^{-i\theta}$, for $0 \le \theta < 2\pi$.
(ii) an infinite dimensional representation π_q defined on a Hilbert space H with orthonormal basis $(e_i)_{i\ge 0}$ by

$$\pi_q(x) e_i = \sqrt{\lambda_{i+1}}\; e_{i+1}, \quad i \ge 0, \tag{16}$$

$$\pi_q(x^*) e_i = \begin{cases} 0 & i = 0, \\ \sqrt{\lambda_i}\; e_{i-1}, & i \ge 1, \end{cases} \tag{17}$$

with $\lambda_i = 1 - q^i$, $i \ge 0$.

It is also shown in [6] that the infinite dimensional representation π_q is faithful. Therefore, $C(D_q)$ has no nontrivial covering. The one-dimensional representations ρ_θ correspond to the classical points, forming a circle, of the quantum disc. Considering $C(S^1)$ as the C^*-algebra generated by a, a^* with relations $aa^* = a^*a = 1$, the embedding of this classical circle into the quantum disc is described by a C^*-homomorphism $\phi_q : C(D_q) \longrightarrow C(S^1)$ defined by $x \mapsto a$. Later we will need

Lemma 2 *The elements $x^k x^{*l}$, $k, l \ge 0$ form a vector space basis of $P(D_q)$. The same is true for the elements $(xx^*)^k x^l$, $k \ge 0$, $l \in \mathbb{Z}$, where $x^{-l} := x^{*l}$, $l > 0$.*

Proof: It is obvious from the relations that every element of $P(D_q)$ can be written as a linear combination of the given elements. Applying the representation π_q to an equation $\sum c_{kl} x^k x^{*l} = 0$ and acting with the zero operator $\pi_q(\sum c_{kl} x^k x^{*l})$ onto suitable basis elements one obtains $c_{kl} = 0$, $\forall k, l$. The linear independence of the second set of elements follows from

$$(xx^*)^k x^l = q^{\frac{1}{2}k(k-1+2l)} x^{k+l} x^{*k} + a^{kl}_{k-1} x^{k+l-1} x^{*k-1} + \ldots + a^{kl}_0 x^l, \quad l \ge 0,$$

$$(xx^*)^k x^{*l} = q^{\frac{1}{2}k(k-1)} x^k x^{*k+l} + a^{k0}_{k-1} x^{k-1} x^{*k-1+l} + \ldots a^{k0}_0 x^{*l}, \quad l > 0,$$

(with some coefficients a^{jk}_l), using a "triangular type" argument. □

The argument proving that the $x^k x^{*l}$ form a basis also shows that π_q is faithful on $P(D_q)$ and that consequently $P(D_q)$ is faithfully embedded in $C(D_q)$.

Lemma 3 *$1 - xx^*$ is not a zero divisor in $P(D_q)$.*

Proof: Assume $(1 - xx^*) \sum_{kl} c_{kl} (xx^*)^k x^l = 0$. Lemma 2 gives the following conditions for the coefficients c_{kl},

$$\begin{aligned} c_{0l} &= 0, \quad \forall l \\ c_{kl} &= c_{k-1,l}, \quad \forall l, k \ge 1 \end{aligned}$$

which lead to $c_{kl} = 0, \quad \forall k, l$. In the same way one proves that $1 - xx^*$ is also not a right zero divisor. $\square$

The C^*-algebra of the unilateral shift is defined as follows: Let H be a Hilbert space with orthonormal basis $(e_i)_{i \geq 0}$. The shift operator $\mathfrak{S} \in B(H)$ is defined by $\mathfrak{S}(e_i) = e_{i+1}$. Its adjoint is given by $\mathfrak{S}^*(e_i) = \begin{cases} e_{i-1} & i > 0 \\ 0 & i = 0. \end{cases}$ $C^*(\mathfrak{S})$ is the C^*-subalgebra of $B(H)$ generated by $\mathfrak{S}$ and $\mathfrak{S}^*$. The symbol map $\sigma : C^*(\mathfrak{S}) \longrightarrow C(S^1)$ is the homomorphism defined by $\sigma(\mathfrak{S}) = a$.

Proposition 15 *$C(D_q)$ is isomorphic to $C^*(\mathfrak{S})$ as C^*-algebra. Under this isomorphism, the symbol map σ corresponds to the embedding of the classical circle, $\phi_q : C(D_q) \longrightarrow C(S^1)$.*

Proof: We use ideas of [5], where this is proved for another one-parameter family of quantum discs. In fact, we prove $\pi_q(C(D_q)) = C^*(\mathfrak{S})$. First, it is easy to see that

$$\pi_q(x) = \mathfrak{S} \sum_{k=0}^{\infty} (\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k}) \mathfrak{S}^k \mathfrak{S}^{*k}, \tag{18}$$

where the series converges in the operator norm. Thus, $\pi_q(C(D_q)) \subset C^*(\mathfrak{S})$. On the other hand, if P_i is the orthogonal projector onto e_i, $\pi_q(x^*)\pi_q(x) = \sum \lambda_{i+1} P_i$ is the spectral resolution of $\pi_q(x^*)\pi_q(x)$, and P_i lies in the C^*-algebra generated by $\pi_q(x^*)\pi_q(x)$, therefore also in $\pi_q(C(D_q))$. The matrix units E_{ij} defined by $E_{ij}(e_k) = \delta_{jk} e_i$ can be written

$$E_{ij} = \begin{cases} (\sqrt{\lambda_{j+1} \cdots \lambda_i})^{-1} \pi_q(x)^{i-j} P_j & i > j, \\ (\sqrt{\lambda_j \cdots \lambda_{i+1}})^{-1} \pi_q(x^*)^{j-i} P_j & i < j, \\ P_i & i = j. \end{cases}$$

Since the E_{ij} generate the ideal $\mathcal{K}$ of compact operators, it follows that $\mathcal{K} \subset \pi_q(C(D_q))$. Moreover, $\mathfrak{S} - \pi_q(x)$ is a weighted shift, $(\mathfrak{S} - \pi_q(x))(e_k) = (1 - \sqrt{\lambda_{k+1}}) e_{k+1}$, where $1 - \sqrt{\lambda_k} \longrightarrow 0$ for $k \longrightarrow \infty$. From the next lemma it follows that $\mathfrak{S} - \pi_q(x) \in \mathcal{K}$, therefore $\mathfrak{S} \in \pi_q(C(D_q))$. For $\sigma \circ \pi_q = \phi_q$ it is sufficient to show $\sigma(\pi_q(x)) = \phi_q(x) = a$, which follows from formula (18) using $\sum_{i=0}^{\infty} (\sqrt{\lambda_{i+1}} - \sqrt{\lambda_i}) = \lim_{k \to \infty} \sqrt{\lambda_k} = 1$. $\square$

Lemma 4 *Let $T \in B(H)$ be a weighted shift,*

$$T(e_i) = t_i e_{i+1},$$

with $t_i \in \mathbb{R}$, $\lim_{i \to \infty} t_i = 0$. Then $T \in \mathcal{K}$.

Proof: $T^*(e_i) = \begin{cases} 0 & i = 0, \\ t_{i-1} e_{i-1} & i > 0. \end{cases}$ Therefore, $T^*T(e_i) = t_i^2 e_i$, and T^*T is a compact operator with spectrum consisting of the isolated eigenvalues t_i^2, $t_i^2 \to 0$. Then $\sqrt{T^*T}$ is also a compact operator with eigenvalues $|t_i|$, and T itself is compact, because its polar decomposition is $T = U\sqrt{T^*T}$, and $\mathcal{K}$ is a two-sided ideal. $\square$

Now, the following proposition is immediate from Proposition 1.2. of [11].

Proposition 16 *For $|\mu| < 1, \ c > 0$, the C^*-algebra $C(S^2_{\mu c})$ of the Podleś sphere is isomorphic to $C(D_p) \oplus_\phi C(D_q) = \{(f, g) \in C(D_p) \oplus C(D_q) | \phi_p(f) = \phi_q(g)\}$, $0 < q, p < 1$.*

The isomorphy holds for any pairs of parameters (μ, c) and (p, q). The images of the generators of $C(S^2_{\mu c})$ under this isomorphism clearly are also generators of $C(D_p) \oplus_\phi C(D_q)$. However, one can also describe $C(D_p) \oplus_\phi C(D_q)$ by means of generators arising naturally from the generators of the two quantum discs via the gluing procedure:

Proposition 17 *Let*

$$P(S^2_{pq\phi}) := \mathbb{C} < f_1, f_{-1}, f_0 > /J_{q,p},$$

where $J_{q,p}$ is the two-sided ideal generated by the relations

$$f_{-1}f_1 - qf_1f_{-1} = (p-q)f_0 + (1-p)1, \tag{19}$$

$$f_0f_1 - pf_1f_0 = (1-p)f_1, \tag{20}$$

$$f_{-1}f_0 - pf_0f_{-1} = (1-p)f_{-1}, \tag{21}$$

$$(1-f_0)(f_1f_{-1} - f_0) = 0. \tag{22}$$

With

$$f_0^* = f_0, \quad f_1^* = f_{-1}, \tag{23}$$

$P(S^2_{pq\phi})$ is a $*$-algebra.
*$P(D_p) \oplus_\phi P(D_q) = \{(f,g) \in P(D_p) \oplus P(D_q) | \phi_p(f) = \phi_q(g)\}$ is isomorphic to the *-algebra $P(S^2_{pq\phi})$.*

Remark: We have the conjecture that $P(S^2_{pq\phi})$ and $P(S^2_{\mu c})$ are not isomorphic as *-algebras.

Proof: Relation (19) is invariant under $*$ whereas (20) and (21) are transformed into each other. (22) is invariant because of $f_1f_{-1}f_0 = f_0f_1f_{-1}$ which follows from (20) and (21).
Next we show $P(D_p) \oplus_\phi P(D_q) \simeq P(S^2_{pq\phi})$. We denote the generators of $P(D_p)$ by x, x^* and those of $P(D_q)$ by y, y^*. Consider the elements $\tilde{f}_0 = (xx^*, 1),\ \tilde{f}_1 = (x, y),\ \tilde{f}_{-1} = (x^*, y^*)$ of $P(D_p) \oplus_\phi P(D_q)$. They fulfill the relations (19) - (22).

Lemma 5 $\tilde{f}_0,\ \tilde{f}_1,\ \tilde{f}_{-1}$ *generate* $P(D_p) \oplus_\phi P(D_q)$.

Proof of the lemma: Use the basis $(xx^*)^k x^l,\ (yy^*)^k y^l,\quad k \geq 0,\ l \in \mathbb{Z}$ of Lemma 2. First we notice that

$$(\sum_{m\geq 0, n\in\mathbb{Z}} c_{mn}(xx^*)^m x^n, \sum_{k\geq 0, l\in\mathbb{Z}} \tilde{c}_{kl}(yy^*)^k y^l) \in P(D_p) \oplus_\phi P(D_q)$$

if and only if

$$\sum_{k\geq 0} c_{kl} = \sum_{k\geq 0} \tilde{c}_{kl}, \quad \forall l. \tag{24}$$

We obtain

$$\begin{aligned}
&(\sum_{m\geq 0, n\in\mathbb{Z}} c_{mn}(xx^*)^m x^n, \sum_{k\geq 0, l\in\mathbb{Z}} \tilde{c}_{kl}(yy^*)^k y^l) = \\
&\sum_{l\in\mathbb{Z}}(\sum_{m\geq 0} c_{ml}(xx^*)^m x^l, \sum_{k\geq 0} \tilde{c}_{kl}(yy^*)^k y^l) = \\
&\sum_{l\in\mathbb{Z}}((\sum_{m\geq 0} c_{ml}(xx^*)^m x^l, \sum_{k\geq 0} \tilde{c}_{kl} y^l) + \\
&(\sum_{m\geq 0} c_{ml} x^l, \sum_{k\geq 0} \tilde{c}_{kl}(yy^*)^k y^l) - (\sum_{m\geq 0} c_{ml} x^l, \sum_{k\geq 0} \tilde{c}_{kl} y^l)) = \\
&\sum_{l\in\mathbb{Z}}(\sum_{m\geq 0} c_{ml}\tilde{f}_0^m \tilde{f}_1^l + \sum_{k\geq 0} \tilde{c}_{kl}(\tilde{f}_1\tilde{f}_{-1} - \tilde{f}_0 + 1)^k \tilde{f}_1^l - \sum_{n\geq 0} c_{nl}\tilde{f}_1^l),
\end{aligned}$$

where we have set $\tilde{f}_1^{-1} = \tilde{f}_{-1}$, and (24) has been used in the last equality. □

By the lemma there exists a surjective homomorphism $F : P(S^2_{pq\phi}) \longrightarrow P(D_p)\bigoplus_\phi P(D_q)$ defined by $F(f_i) := \tilde{f}_i$. Let $p_1 : P(D_p)\bigoplus_\phi P(D_q) \longrightarrow P(D_p)$ and $p_2 : P(D_p)\bigoplus_\phi P(D_q) \longrightarrow P(D_q)$ be the restrictions of the first and second projections. By definition $ker\, p_1 \bigcap ker\, p_2 = 0$. Let $\pi_1 := p_1 \circ F$ and $\pi_2 := p_2 \circ F$, i. e. $\pi_1(f_0) = xx^*,\ \pi_1(f_1) = x,\ \pi_2(f_0) = 1,\ \pi_2(f_1) = y$. F is an isomorphism if $ker\pi_1 \bigcap ker\pi_2 = 0$.

First we describe $ker\pi_2$. Every element $a \in P(S^2_{pq\phi})$ can be written in the form

$$a = \sum_{k\in\mathbb{Z};m,n\geq 0} c_{mnk}(f_1f_{-1})^m f_0^n f_1^k,$$

where $f_1^{-1} = f_{-1}$. Applying π_2 to $a \in ker\,\pi_2$, it follows that

$$\sum_{k\in\mathbb{Z};m,n\geq 0} c_{mnk}(yy^*)^m y^k = 0,$$

and one obtains the condition $\sum_{n\geq 0} c_{mnk} = 0, \quad \forall m, k$. Thus, we have the identity

$$\begin{aligned}\sum_{n\geq 0} c_{mnk}(f_1f_{-1})^m f_0^n f_1^k &= \sum_{n\geq 1}\sum_{s=1}^{n} c_{mnk}(f_1f_{-1})^m (f_0^s - f_0^{s-1}) f_1^k \\ &= \sum_{n\geq 1}\sum_{s=1}^{n} c_{mnk}(f_1f_{-1})^m (f_0 - 1) f_0^{s-1} f_1^k.\end{aligned}$$

In view of $f_1f_{-1}f_0 = f_0f_1f_{-1}$, this means that every element $a \in ker\pi_2$ can be written in the form

$$a = (1 - f_0) \sum_{k\in\mathbb{Z};m,n\geq 0} a_{mnk}(f_1f_{-1})^m f_0^n f_1^k. \tag{25}$$

Assume now $a \in ker\pi_1 \bigcap ker\pi_2$. Applying π_1 to a one obtains

$$(1 - xx^*) \sum_{k\in\mathbb{Z};m,n\geq 0} a_{mnk}(xx^*)^m (xx^*)^n x^k = 0,$$

which yields, since $1 - xx^*$ is not a zero divisor, the following condition for the coefficients a_{mnk}:

$$\sum_{n=0}^{l} a_{l-n,n,k} = 0,\ \forall l \geq 0, k. \tag{26}$$

This leads to

$$\begin{aligned} & (1 - f_0) \sum_{k\in\mathbb{Z},m,n\geq 0} a_{mnk}(f_1f_{-1})^m f_0^n f_n^k \\ = \ & (1 - f_0) \sum_{k\in\mathbb{Z}}\sum_{l\geq 0}\sum_{n=0}^{l} a_{l-n,n,k}(f_1f_{-1})^{l-n} f_0^n f_n^k \\ = \ & (1 - f_0) \sum_{k\in\mathbb{Z}}\sum_{l\geq 0}\sum_{n=1}^{l}\sum_{s=1}^{n} a_{l-n,n,k}((f_1f_{-1})^{l-s} f_0^s - (f_1f_{-1})^{l-s+1} f_0^{s-1}) f_1^k \\ = & -(1 - f_0)(f_1f_{-1} - f_0) \sum_{k\in\mathbb{Z}}\sum_{l\geq 0}\sum_{n=1}^{l}\sum_{s=1}^{n} a_{l-n,n,k}(f_1f_{-1})^{l-s} f_0^{s-1} f_1^k = 0. \end{aligned} \tag{27}$$

(26) was used in the second equality. Thus $ker\,\pi_1 \bigcap ker\,\pi_2 = 0$, and F is an isomorphism. $\square$

Note that the computation leading to (27) can also be used to show that $ker\pi_1$ is generated by $f_1f_{-1} - f_0$, since the application of π_1 to a general element of $ker\pi_1$ leads to (27) without the factor $1 - f_0$.

Proposition 18 *Let ρ be a homomorphism of the *-algebra $P(S^2_{pq\phi})$ into the *-algebra $B(H)$ of bounded operators on a Hilbert space H.*
Then ker $(1-\rho(f_0))$ and ker $\rho(f_1f_{-1}-f_0)$ are closed subspaces invariant under all representation operators. H can be decomposed into the orthogonal direct sum

$$H = ker(1 - \rho(f_0)) \oplus ker(\rho(f_1f_{-1} - f_0)|_{(ker(1-\rho(f_0)))^\perp}). \tag{28}$$

Proof: The invariance of the kernels is a direct consequence of the relations. The kernels are closed since they belong to bounded operators. Thus, H can be decomposed into the orthogonal direct sum $H = ker(1 - \rho(f_0)) \oplus ker(1 - \rho(f_0))^\perp$ of closed invariant subspaces. In turn, $ker(1-\rho(f_0))^\perp$ can be decomposed as $ker(\rho(f_1f_{-1}-f_0)|_{(ker(1-\rho(f_0)))^\perp}) \oplus H^c$, with another invariant subspace H^c. The restrictions to H^c of both $1 - \rho(f_0)$ and $\rho(f_1f_{-1} - f_0)$ are injective. Thus, if $\psi \in H^c, \psi \neq 0$, it follows that $(1 - \rho(f_0))\rho(f_1f_{-1} - f_0)(\psi) \neq 0$, which contradicts relation (22). □

In the restriction of ρ to $ker(1 - \rho(f_0))$ we have $\rho(f_0) = 1$, and the relations reduce to

$$\rho(f_{-1})\rho(f_1) - q\rho(f_1)\rho(f_{-1}) = (1 - q)1,$$

which are the relations of a quantum disc with parameter q. On the complement $ker(\rho(f_1f_{-1} - f_0)|_{(ker(1-\rho(f_0)))^\perp})$ we have $\rho(f_0) = \rho(f_1f_{-1})$, and the relation (19) reduces to

$$\rho(f_{-1})\rho(f_1) - p\rho(f_1)\rho(f_{-1}) = (1 - p)1,$$

i. e. the relations of a quantum disc with parameter p. (20) and (21) follow from these relations, whereas (22) is satisfied trivially. Using the results of [6] for quantum discs, we obtain

Proposition 19 *The following is a complete list of bounded irreducible *-representations of $P(S^2_{pq\phi})$:*
1. A representation in a Hilbert space H with orthonormal basis $(e_i)_{i=0,1,...}$

$$\rho_1(f_0)e_i = \begin{cases} 0 & i = 0 \\ \lambda_i e_i & i > 0 \end{cases} \tag{29}$$

$$\rho_1(f_1)e_i = \sqrt{\lambda_{i+1}}\ e_{i+1}, \quad i \geq 0, \tag{30}$$

$$\rho_1(f_{-1})e_i = \begin{cases} 0 & i = 0, \\ \sqrt{\lambda_i}\ e_{i-1}, & i \geq 1, \end{cases} \tag{31}$$

with $\lambda_i = 1 - p^i$, $i \geq 0$.
2. A representation in H given by

$$\rho_2(f_0)e_i = e_i, \tag{32}$$

$$\rho_2(f_1)e_i = \sqrt{\lambda'_{i+1}}\ e_{i+1}, \quad i \geq 0, \tag{33}$$

$$\rho_2(f_{-1})e_i = \begin{cases} 0 & i = 0, \\ \sqrt{\lambda'_i}\ e_{i-1}, & i \geq 1, \end{cases} \tag{34}$$

with $\lambda'_i = 1 - q^i$, $i \geq 0$.
3. A one parameter family of one dimensional representations given by

$$\rho_\theta(f_0) = 1, \tag{35}$$

$$\rho_\theta(f_1) = e^{i\theta}, \tag{36}$$

$$\rho_\theta(f_{-1}) = e^{-i\theta}, \tag{37}$$

where $0 \leq \theta < 2\pi$.
Moreover, $\|\rho(f_1)\| = \|\rho(f_{-1})\| = \|\rho(f_0)\| = 1$ *for any *-representation of* $P(S^2_{pq\phi})$ *in bounded operators.*

Denoting by Rep_b the set of *-representations of $P(S^2_{pq\phi})$ in bounded operators, it follows that for each $a \in P(S^2_{pq\phi})$ exists $\|a\| := \sup_{\rho \in Rep_b} \|\rho(a)\| < \infty$.

Proposition 20 *(i)* $\|.\|$ *is a C*-norm on* $P(S^2_{pq\phi})$.
(ii) $\rho_1 \oplus \rho_2$ *is a faithful representation of* $P(S^2_{pq\phi})$.
(iii) $\{f_1^k f_0 f_{-1}^l, f_1^k f_{-1}^l | k,l = 0,1,\ldots\}$ *is a vector space basis of* $P(S^2_{pq\phi})$.

Proof: As a first step, one shows that the vectors (iii) form a linear generating system. For this, one first shows inductively that the monomials $f_1^k f_0^l f_{-1}^m,\ k,l,m = 0,1,\ldots$ form a linear generating system. Then one uses (22) to reduce the power of f_0.
Let now $a = \sum_{k,l=0,1\ldots}(a_{kl} f_1^k f_0 f_{-1}^l + b_{kl} f_1^k f_{-1}^l)$, and assume $\rho_1 \oplus \rho_2(a) = 0$. Then we have

$$\rho_1(a)e_0 = \sum_k (a_{k0}\rho_1(f_1)^k \rho_1(f_0)e_0 + b_{k0}\rho_1(f_1)^k e_0) = \sum_k b_{k0}\sqrt{\lambda_k \cdots \lambda_1} e_k = 0,$$

i. e. $b_{k0} = 0$ for all k. Thus,

$$\rho_2(a)e_0 = \sum_k (a_{k0}\sqrt{\lambda'_k \cdots \lambda'_1} e_k + b_{k0}\sqrt{\lambda'_k \cdots \lambda'_1} e_k) = 0$$

gives $a_{k0} = 0$ for all k.

Assume now that

$$a_{kl} = b_{kl} = 0,\ \forall k$$

is shown for $l \leq i$. Then

$$\rho_1(a)e_{i+1} = \sum_k (a_{k,i+1}\rho_1(f_1)^k \rho_1(f_0)\sqrt{\lambda_1 \cdots \lambda_{i+1}} e_0 + b_{k,i+1}\rho_1(f_1)^k \sqrt{\lambda_1 \cdots \lambda_{i+1}} e_0) =$$

$$= \sum_k b_{k,i+1}\sqrt{\lambda_k \cdots \lambda_1^2 \cdots \lambda_{i+1}} e_k = 0,$$

i. e. $b_{k,i+1} = 0$. $\forall k$. From

$$\rho_2(a)e_{i+1} = \sum_k (a_{k,i+1}\sqrt{\lambda'_k \cdots {\lambda'_1}^2 \cdots \lambda'_{i+1}} e_k + b_{k,i+1}\sqrt{\lambda'_k \cdots {\lambda'_1}^2 \cdots \lambda'_{i+1}} e_k) = 0$$

now also follows $a_{k,i+1} = 0$, $\forall k$. This proves the proposition. □

Definition 12 $C(S^2_{pq\phi})$ *is the closure of* $P(S^2_{pq\phi})$ *in the norm* $\|.\|$.

Proposition 21 $C(S^2_{pq\phi})$ *is C*-isomorphic to* $C(S^2_{\mu c})$ *for* $c > 0$.

Proof: It is sufficient to prove that $C(S^2_{pq\phi})$ is isomorphic to $C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S})$. Using that reduced atomic representations are faithful, it is enough to see that one can choose for both algebras one element from every equivalence class of irreducible representations in such a way that there is a bijection between the resulting sets of representations, and that the images of corresponding

representations are isomorphic as C*-algebras. It follows again from Proposition 1.2 of [11] that the irreducible representations of $C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S})$ are up to equivalence p_1', p_2' and $p_\theta \circ \sigma$, where p_1', p_2' are the restrictions of the first and second projections to $C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S})$, and p_θ is the evaluation at $e^{i\theta}$, $p_\theta(f) = f(e^{i\theta})$ for $f \in C(S^1)$. Indeed, p_1', p_2', $p_\theta \circ \sigma$ correspond under the isomorphism $(\pi_+, \pi_-) : C(S^2_{\mu c}) \longrightarrow C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S})$ to the representations π_+, π_-, π_θ of $C(S^2_{\mu c})$ (see [9]) respectively. The representations of $C(S^2_{pq\phi})$ corresponding to p_1', p_2', $p_\theta \circ \sigma$ are now ρ_1, ρ_2, ρ_θ respectively. The equality of the corresponding images is trivial for the one-dimensional representations and follows for the others with the same arguments as in the proof of Proposition 15. □

Note that it would have been sufficient to use only ρ_1, ρ_2 and p_1', p_2' in the above proof, because $\rho_1 \oplus \rho_2$ and $p_1' \oplus p_2' = id$ are already faithful representations. For $\rho_1 \oplus \rho_2$ this follows from $\rho_\theta = p_\theta \circ \sigma \circ \rho_i$, $i = 1, 2$. Moreover, there are the equalities $p_i' \circ (\pi_p \oplus \pi_q) \circ F = \rho_i$, $i = 1.2$, and $p_\theta \circ \sigma \circ (\pi_p \oplus \pi_q) \circ F = \rho_\theta$, which mean that F extends to a C*-isomorphism $C(S^2_{pq\phi}) \longrightarrow C(D_p) \oplus_\phi C(D_q)$.

In order to determine an underlying "space" of $C(S^2_{pq\phi})$, we look for the spectra of generators. First we introduce instead of f_1, f_{-1} the selfadjoint elements $f_+ = \frac{1}{2}(f_1 + f_{-1})$, $f_- = \frac{1}{2}i(f_1 - f_{-1})$. In terms of f_+ and f_- the relations (19) - (22) are

$$(1-q)(f_+^2 + f_-^2) + (1+q)i(f_- f_+ - f_+ f_-) = (p-q)f_0 + (1-p)1, \tag{38}$$

$$f_0 f_+ - p f_+ f_0 - i(f_0 f_- - p f_- f_0) = (1-p)(f_+ - i f_-), \tag{39}$$

$$f_+ f_0 - p f_0 f_+ + i(f_- f_0 - p f_0 f_-) = (1-p)(f_+ + i f_-), \tag{40}$$

$$(1 - f_0)(f_+^2 + f_-^2 + i(f_+ f_- - f_- f_+) - f_0) = 0. \tag{41}$$

Putting here $p = q = 1$, (38) - (40) just mean commutativity of the algebra, whereas the geometric counterpart of (41) is the union of the plane $f_0 = 1$ and the cone $f_+^2 + f_-^2 = f_0$ in f_0, f_+, f_--space. For $p, q \neq 1$, f_+ and f_- act in the irreducible representations as follows :

$$\rho_1(f_+)e_k = \frac{1}{2}(\sqrt{\lambda_{k+1}}e_{k+1} + \sqrt{\lambda_k}e_{k-1}), \tag{42}$$

$$\rho_1(f_-)e_k = \frac{1}{2}i(\sqrt{\lambda_{k+1}}e_{k+1} - \sqrt{\lambda_k}e_{k-1}), \tag{43}$$

$\rho_2(f_\pm)$ obey (42) and (43) with λ_k' in place of λ_k, and

$$\rho_\theta(f_+) = \cos\theta, \quad \rho_\theta(f_-) = \sin\theta. \tag{44}$$

It follows that

$$\rho_1(f_+^2 + f_-^2)e_i = (1 - \frac{1}{2}(p^i + p^{i+1}))e_i$$

and

$$\rho_2(f_+^2 + f_-^2)e_i = (1 - \frac{1}{2}(q^i + q^{i+1}))e_i,$$

whereas $\rho_{1,2}(f_\pm)$ are Jacobi "matrices" with continuous spectra. So one may draw the following picture in f_0, f_+, f_--space, assigning to every "eigenstate" e_i of $\rho_{1,2,\theta}(f_0)$ the possible values of a "measurement" of f_0, f_+, f_-:

ρ_θ, $0 \leq \theta < 2\pi$ give rise to the circle $f_0 = 1$, $f_+^2 + f_-^2 = 1$. ρ_2 leads to circles $f_0 = 1$, $f_+^2 + f_-^2 = 1 - \frac{1}{2}(q^i + q^{i+1})$, and ρ_1 to circles $f_0 = 1 - p^i$, $f_+^2 + f_-^2 = 1 - \frac{1}{2}(p^i + p^{i+1})$. The union of all these circles may be considered as a discretized version of the top of the cone arising in the classical case, i. e. the set $\{(f_0, f_+, f_-) | f_0 = 1, f_+^2 + f_-^2 \leq 1\} \cup \{(f_0, f_+, f_-) | 0 \leq f_0 \leq 1, f_+^2 + f_-^2 = f_0\}$.

As a consequence of the considerations in the proof of Proposition 17 the homomorphisms $\pi_{1,2} : P(S^2_{pq\phi}) \longrightarrow P(D_{p,q})$ define a covering $(ker\, \pi_1, ker\, \pi_2)$, which is just the transport of the covering $(ker\, p_1, ker\, p_2)$ of $P(D_p) \oplus_\phi P(D_q)$ under the isomorphism F.

Our task is now the construction of a differential calculus over $P(S^2_{pq\phi})$ adapted to the covering $(ker\, \pi_1, ker\, \pi_2)$. According to Proposition 11, such a differential calculus is uniquely determined by differential calculi over $P(D_p) \simeq P(S^2_{pq\phi})/ker\, \pi_1$ and $P(D_q) \simeq P(S^2_{pq\phi})/ker\, \pi_2$. We choose $\Gamma(P(D_p)) = \Omega(P(D_p))/J(P(D_p))$, where $J(P(D_p))$ is generated by the elements:

$$\begin{array}{rcl} x(dx) & - & p^{-1}(dx)x \\ x^*(dx^*) & - & p(dx^*)x^* \\ x(dx^*) & - & p^{-1}(dx^*)x \\ x^*(dx) & - & p(dx)x^*, \end{array}$$

analogously for $P(D_q)$. These differential calculi were considered in [12]. Obviously, they are $*$-differential calculi.

Lemma 6 *(i) dx and dx^* form a left and right $P(D_q)$-module basis of $\Gamma^1(P(D_q))$.*
(ii) $dxdx^*$ is a left and right $P(D_q)$-module basis of $\Gamma^2(P(D_q))$.
(iii) $\Gamma^n(P(D_q)) = 0$ for $n \geq 3$.

This is proved in the appendix.

For $p \neq q$ the differential calculus $\Gamma(P(S^2_{pq\phi}))$ obtained by gluing together $\Gamma(P(D_p))$ and $\Gamma(P(D_q))$ has the following form:

$$\begin{array}{rcl} \Gamma^0(P(S^2_{pq\phi})) & = & P(S^2_{pq\phi}) \\ \Gamma^n(P(S^2_{pq\phi})) & = & \Gamma^n(P(D_q)) \bigoplus \Gamma^n(P(D_p)), \quad \forall n > 0. \end{array}$$

This follows from formula (14), which in our case reads $J(P(S^1)) = \phi_{p\Omega}(J(P(D_p)) + \phi_{q\Omega}(J(P(D_q))$. We get in the differential ideal $J(P(S^1))$ elements of the form

$$\begin{array}{rcl} a(da) - p^{-1}(da)a & = & \phi_{p\Omega}(x(dx) - p^{-1}(dx)x) \\ a(da) - q^{-1}(da)a & = & \phi_{q\Omega}(y(dy) - q^{-1}(dy)y), \end{array}$$

where a is the generator of $P(S^1)$, which leads to $(q^{-1} - p^{-1})da \in J(P(S^1))$ and $da = 0$ in $\Gamma(P(S^1))$. In the same way follows $da^* = 0$ in $\Gamma(P(S^1))$, which means $\Gamma^n(P(S^1)) = 0$, $\forall n > 0$. So, there is no gluing in all degrees $n > 0$.

In the case $q = p$ the differential calculus on $P(S^2_{pq\phi})$ in higher degree than zero is not simply the direct sum of the differential calculi on the quantum discs. The differential ideal $J(P(S^2_{qq\phi}))$ defining the differential calculus $\Gamma(P(S^2_{qq\phi}))$ can be written in terms of the generators f_1, f_{-1} and f_0 as follows:

Proposition 22 *Let $\Gamma(P(S^2_{qq\phi})) := \Omega(P(S^2_{qq\phi}))/J(P(S^2_{qq\phi}))$, where the differential ideal $J(P(S^2_{qq\phi}))$ is generated by the elements*

$$f_1(df_1) - q^{-1}(df_1)f_1, \quad f_{-1}(df_{-1}) - q(df_{-1})f_{-1}, \tag{45}$$
$$f_1(df_{-1}) - q^{-1}(df_{-1})f_1, \quad f_{-1}(df_1) - q(df_1)f_{-1}, \tag{46}$$
$$f_0(df_1) - (df_1)f_0, \quad f_0(df_{-1}) - (df_{-1})f_0 \tag{47}$$
$$(df_0)(f_1f_{-1} - f_0), \quad (f_1f_{-1} - f_0)df_0, \tag{48}$$

and

$$(1-q)df_0df_{-1} - qf_{-1}df_0df_0 \tag{49}$$
$$(1-q)df_1df_0 - qf_1df_0df_0 \tag{50}$$
$$(1-f_0)((1-q)df_1df_{-1} - df_0df_0) \tag{51}$$
$$(f_1f_{-1} - f_0)df_0df_0. \tag{52}$$

Then the homomorphisms π_1 *and* π_2 *are differentiable and*

$$ker\pi_{1_\Gamma} \bigcap ker\pi_{2_\Gamma} = \{0\}, \tag{53}$$

i.e. $\Gamma(P(S^2_{qq\phi}))$ *is the unique differential calculus such that* $(\Gamma(P(S^2_{qq\phi})), (\Gamma(P(D_q)), \Gamma(P(D_q))))$ *is adapted to* $(P(S^2_{qq\phi}), (ker\,\pi_1, ker\,\pi_2))$ *according to Proposition 11.*

Proof: One shows easily that $\pi_{1_\Omega}(J(P(S^2_{qq\phi}))) \subset J(P(D_q))$ and $\pi_{2_\Omega}(J(P(S^2_{qq\phi}))) \subset J(P(D_q))$, i.e. π_1 and π_2 are differentiable.

First let us prove the assertion (53) for the first degree $\Gamma^1(P(S^2_{qq\phi}))$, i. e. $(ker\pi_{1_\Gamma})^1 \bigcap (ker\pi_{2_\Gamma})^1 = \{0\}$. Using Proposition 20, (iii), and (45), (46) and (47) one finds that every element $\gamma \in \Gamma(P(S^2_{qq\phi}))$ can be written in the form

$$\begin{aligned}
\gamma &= \sum_{k,l\geq 0} (a_{k,l} f_1^k f_0 f_{-1}^l + b_{k,l} f_1^k f_{-1}^l) df_1 \\
&+ \sum_{k,l\geq 0} (c_{k,l} f_1^k f_0 f_{-1}^l + d_{k,l} f_1^k f_{-1}^l) df_{-1} \\
&+ \sum_{k,l\geq 0} (e_{k,l} f_1^k f_0 f_{-1}^l + g_{k,l} f_1^k f_{-1}^l) df_0.
\end{aligned}$$

Assuming $\gamma \in ker\pi_{2_\Gamma}$ one obtains

$$\begin{aligned}
\pi_{2_\Gamma}(\gamma) &= \sum_{k,l\geq 0} (a_{k,l} y^k y^{*l} + b_{k,l} y^k y^{*l}) dy \\
&+ \sum_{k,l\geq 0} (c_{k,l} y^k y^{*l} + d_{k,l} y^k y^{*l}) dy^* = 0.
\end{aligned}$$

Using the bimodule basis $\{dy, dy^*\}$ of $\Gamma(P(D_q))$ and the vector space basis $\{y^k y^{*l} | k,l = 0,1,\ldots\}$ of $P(D_q)$ one obtains $a_{k,l} = -b_{k,l}$ and $c_{k,l} = -d_{k,l}$. It follows that an element $\gamma \in ker\pi_{2_\Gamma}$ can be written in the form

$$\begin{aligned}
\gamma &= \sum_{k,l\geq 0} a_{k,l} f_1^k (f_0 - 1) f_{-1}^l df_1 \\
&+ \sum_{k,l\geq 0} c_{k,l} f_1^k (f_0 - 1) f_{-1}^l df_{-1} \\
&+ \sum_{k,l\geq 0} (e_{k,l} f_1^k f_0 f_{-1}^l + g_{k,l} f_1^k f_{-1}^l) df_0.
\end{aligned}$$

The relations in the algebra give the identities

$$\begin{aligned}
(f_0 - 1) f_i &= q^i f_i (f_0 - 1),\ i = -1, 0, 1, \\
f_{-1}^n f_1 &= q^n f_1 f_{-1}^n + (1 - q^n) f_{-1}^{n-1},\ n > 0.
\end{aligned}$$

From (48) follows

$$\begin{aligned} f_1f_{-1}df_0 &= f_0df_0, \\ (f_0-1)f_{-1}df_1 &= q(f_0-1)df_0 - q(f_0-1)f_1df_{-1}. \end{aligned}$$

The last four equations, together with (21), (22) and (46) now give the following two identities:

$$\begin{aligned} f_0f_{-1}^l df_0 &= f_1f_{-1}^{l+1}df_0 \\ &+ (q^{-l}-1)f_{-1}^l df_0 + l(1-q^{-1})f_{-1}^l df_0, \\ (f_0-1)f_{-1}^l df_1 &= q(f_0-1)f_{-1}^{l-1}df_0 \\ &- q^{l+1}f_1(f_0-1)f_{-1}^{l-1}df_{-1} - q^2(1-q^{l-1})(f_0-1)f_{-1}^{l-2}df_{-1},\ l>0. \end{aligned}$$

It follows that $\gamma \in ker\pi_{2_\Gamma}$ can be written in the form

$$\begin{aligned} \gamma &= \sum_{k\geq 0} \tilde{a}_k f_1^k(f_0-1)df_1 \\ &+ \sum_{k,l\geq 0} \tilde{c}_{k,l}(f_0-1)f_1^k f_{-1}^l df_{-1} + \sum_{k,l\geq 0} \tilde{g}_{k,l} f_1^k f_{-1}^l df_0. \end{aligned}$$

Assuming $\gamma \in ker\pi_{1_\Gamma} \bigcap ker\pi_{2_\Gamma}$ one obtains

$$\begin{aligned} \pi_{1_\Gamma}(\gamma) &= \sum_{k\geq 0} \tilde{a}_k x^k(xx^*-1)dx \\ &+ \sum_{k,l\geq 0} \tilde{c}_{k,l}(xx^*-1)x^k x^{*l}dx^* + \sum_{k,l\geq 0} \tilde{g}_{k,l} x^k x^{*l} d(xx^*) = 0. \end{aligned}$$

The left coefficient of dx is

$$\sum_{k\geq 0}(\tilde{a}_k x^{k+1}x^* - \tilde{a}_k x^k) + q^{-1}\sum_{k,l\geq 0} q^{-1}\tilde{g}_{k,l}x^k x^{*l+1} = 0,$$

which gives $\tilde{a}_k = 0,\ \forall k$ and $\tilde{g}_{k,l} = 0,\ \forall k,l$. This leads to

$$\pi_{1_\Gamma}(\gamma) = \sum_{k,l\geq 0} \tilde{c}_{k,l}(xx^*-1)x^k x^{*l} = 0.$$

Since xx^*-1 is not a zero divisor, it follows that $\tilde{c}_{k,l} = 0\ \forall k,l$, i.e. $\gamma = 0$.

Now, let us prove the assertion for $\Gamma^2(P(S^2_{qq\phi}))$.
Applying d to (45)-(48) one obtains in $\Gamma(P(S^2_{qq\phi}))$, besides (49)-(52), the following relations

$$\begin{aligned} &df_1df_1 = 0,\quad df_{-1}df_{-1} = 0,\quad df_{-1}df_1 = -qdf_1df_{-1}, \\ &df_0df_1 = -df_1df_0,\quad df_{-1}df_0 = -df_0df_{-1}, \end{aligned}$$

and one can see that $\Gamma^2(P(S^2_{qq\phi}))$ is generated by the elements df_1df_{-1} and df_0df_0 as a $P(S^2_{qq\phi})$-bimodule. Let us consider a general element $\gamma \in \Gamma^2(P(S^2_{qq\phi}))$,

$$\gamma = adf_1df_{-1} + bdf_0df_0.$$

Applying π_{2_Γ} to $\gamma \in ker\pi_{2_\Gamma}$ gives

$$\pi_2(a)dxdx^* = 0,$$

and it follows $a \in ker\pi_2$(Lemma 6, (ii)). $ker\pi_2$ is generated by the element $1-f_0$ (see (25)), and because of relation (51) $ker\pi_{2_\Gamma}$ is generated by df_0df_0. Now assume

$$\gamma = \tilde{b}df_0df_0 \in ker\pi_{1_\Gamma}\bigcap ker\pi_{2_\Gamma}.$$

It follows that

$$\begin{aligned}
\pi_{1_\Gamma}(\gamma)=\pi_1(\tilde{b})d(xx^*)d(xx^*) &= \pi_1(\tilde{b})(x(dx^*)(dx)x^*+(dx)x^*x^*(dx))\\
&= \pi_1(\tilde{b})(x^*xdxdx^*+xx^*dx^*dx)\\
&= \pi_1(\tilde{b})(x^*xdxdx^*-qxx^*dxdx^*)\\
&= \pi_1(\tilde{b})(1-q)dxdx^*=0,
\end{aligned}$$

and this leads to $\tilde{b}\in ker\pi_1$.

As noted above $ker\pi_1$ is generated by the element $f_1f_{-1}-f_0$. It is immediate from (19)-(22) that $(f_1f_{-1}-f_0)f_i=q^if_i(f_1f_{-1}-f_0),\quad i=\pm1,0$. This and (52) gives $\gamma=0$, i.e. $(ker\pi_{1_\Gamma})^2\bigcap(ker\pi_{2_\Gamma})^2=0$.

Finally, $\Gamma^n(P(S^2_{qq\phi}))=0,\quad\forall n>2$, i.e. $J^n(P(S^2_{qq\phi}))=\Omega^n(P(S^2_{qq\phi})),\quad\forall n>2$, since $\Gamma^n(P(D_q))=0,\quad\forall n>2$. This can also be obtained directly from the generators of $J(P(S^2_{qq\phi}))$ ((45)- (48) and (49)- (52)), and we need no additional relations for higher degrees. $\square$

$\Gamma(P(S^2_{qq\phi}))$ is a $*$-differential calculus:

Obviously, the homomorphisms π_1 and π_2 are $*$-homomorphisms. We know that the universal differential calculus $\Omega(P(S^2_{qq\phi}))$ is a $*$-differential calculus, and one easily verifies that the homomorphisms $\pi_{1_{\Omega\to\Gamma}}$ and $\pi_{2_{\Omega\to\Gamma}}$ are $*$-homomorphisms. It follows that the kernels of these homomorphisms are $*$-ideals and also the intersection of these kernels is a $*$-ideal. This ideal is just the differential ideal defining the differential calculus $\Gamma(P(S^2_{qq\phi}))$, i.e. there exists a $*$-structure on $\Gamma(P(S^2_{qq\phi}))$, such that the quotient map $\pi_{\Omega,\Gamma}:\Omega(P(S^2_{qq\phi}))\longrightarrow\Gamma(P(S^2_{qq\phi}))$ satisfies

$$\pi_{\Omega,\Gamma}\circ*=*\circ\pi_{\Omega,\Gamma}.$$

Finally, the differential calculus $\Gamma(P(S^2_{qq\phi}))$ is also $U_{q^{1/2}}(sl_2)$-covariant:

First we recall the meaning of the covariance of the differential calculus over the quantum disc with respect to the action of $U_{q^{1/2}}(sl_2)$ and refer for more details to [12].

The Hopf algebra $U_{q^{1/2}}(sl_2)$ is the algebra generated by $K^{\pm1}$, E, F with the relations

$$\begin{aligned}
&KK^{-1}=K^{-1}K=1,\quad K^{\pm1}E=q^{\pm1}EK^{\pm1},\quad K^{\pm1}F=q^{\mp1}FK^{\pm1}\\
&EF-FE=(K-K^{-1})/(q^{1/2}-q^{-1/2}).
\end{aligned}$$

The comultiplication Δ, the counit ε and the antipode S are defined as follows

$$\begin{aligned}
&\Delta(E)=E\otimes1+K\otimes E,\ \Delta(F)=F\otimes K^{-1}+1\otimes F,\\
&\Delta(K^{\pm1})=K^{\pm1}\otimes K^{\pm1},\\
&\varepsilon(E)=\varepsilon(F)=0,\quad\varepsilon(K^{\pm1})=1,\\
&S(E)=-K^{-1}E,\quad S(F)=-FK,\quad S(K^{\pm1})=K^{\mp1}.
\end{aligned}$$

There exists an action $\cdot:U_{q^{1/2}}(sl_2)\times P(D_q)\longrightarrow P(D_q)$, which means

$$\begin{aligned}
h\cdot1 &= \varepsilon(h)1,\quad\forall h\in U_{q^{1/2}}(sl_2)\\
1\cdot a &= a,\quad\forall a\in P(D_q)\\
h\cdot(ab) &= \sum(h_1\cdot a)(h_2\cdot b),\quad\forall h\in U_{q^{1/2}}(sl_2),\quad a,b\in P(D_q)\\
h\cdot(g\cdot a) &= (hg)\cdot a,\quad\forall h,g\in U_{q^{1/2}}(sl_2),\quad a,\in P(D_q),
\end{aligned}$$

defined by

$$\begin{aligned}
&K^{\pm1}\cdot x:=q^{\pm1}x,\quad F\cdot x:=q^{1/4}1,\quad E\cdot x:=-q^{1/4}x^2\\
&K^{\pm}\cdot x^*:=q^{\mp1}x^*,\quad F\cdot x^*:=-q^{5/4}x^{*2},\quad E\cdot x^*:=q^{-3/4}1.
\end{aligned}$$

It was shown in [12] that this action can be extended to the differential algebra $\Gamma(P(D_q))$, which means that there exists $\cdot : U_{q^{1/2}}(sl_2) \times \Gamma(P(D_q)) \longrightarrow \Gamma(P(D_q))$ defined by

$$h \cdot da = d(h \cdot a),$$

and

$$h \cdot \gamma\sigma = \sum (h_1 \cdot \gamma)(h_2 \cdot \sigma), \quad h \in U_{q^{1/2}}(sl_2), \quad \gamma, \sigma \in \Gamma(P(D_q)).$$

It turns out that this action is compatible with the gluing procedure, i.e. there is an action $\cdot : U_{q^{1/2}}(sl_2) \times P(S^2_{qq\phi}) \longrightarrow P(S^2_{qq\phi})$ given by

$$h \cdot (a, b) := (h \cdot a, h \cdot b), \quad h \in U_{q^{1/2}}(sl_2), \quad (a, b) \in P(S^2_{qq\phi}).$$

On the generators f_1, f_{-1} and f_0 we have

$$\begin{array}{l} K^{\pm 1} \cdot f_1 = q^{\pm 1} f_1, \quad F \cdot f_1 = q^{1/4} 1, \quad E \cdot f_1 = -q^{1/4} f_1^2, \\ K^{\pm 1} \cdot f_{-1} = q^{\mp 1} f_{-1}, \quad F \cdot f_{-1} = -q^{5/4} f_{-1}^2, \quad E \cdot f_{-1} = q^{-3/4} 1, \\ K^{\pm 1} \cdot f_0 = f_0, \quad F \cdot f_0 = q^{5/4}(f_{-1} - f_0 f_{-1}), \quad E \cdot f_0 = q^{1/4}(f_1 - f_1 f_0). \end{array}$$

The extension to $\Gamma(P(S^2_{qq\phi}))$ is obvious. The projections π_1 and π_2 intertwine the actions of $U_{q^{1/2}}(sl_2)$ on $P(S^2_{qq\phi})$ and on the two copies of $P(D_q)$. This property extends to the universal differential calculus $\Omega(P(S^2_{qq\phi}))$, and we even have

$$\pi_{i_{\Omega \to \Gamma}}(h \cdot \gamma) = h \cdot \pi_{i_{\Omega \to \Gamma}}(\gamma),$$

which means that the kernels of the homomorphisms $\pi_{i_{\Omega \to \Gamma}}$ are invariant under the action $\cdot$. Thus, also $\ker \pi_{1_{\Omega \to \Gamma}} \bigcap \ker \pi_{2_{\Omega \to \Gamma}}$ is invariant under the action $\cdot$. This intersection is just our differential ideal $J(P(S^2_{qq\phi}))$, and it follows that one can extend the action $\cdot$ on $P(S^2_{qq\phi})$ to the differential calculus $\Gamma(P(S^2_{qq\phi}))$, i.e. $\Gamma(P(S^2_{qq\phi}))$ is covariant.

It is also easy to show that $P(S^2_{qq\phi})$ and $\Gamma(P(S^2_{qq\phi}))$ are covariant with respect to $U_{q^{1/2}}(su(1,1))$ as $*$-algebras. (cf.[12])

5 Appendix

The purpose of this appendix is to prove Lemma 6.

(i) We have to show that from $adx + bdx^* = 0,\ a, b \in P(D_q)$ follows $a = 0$ and $b = 0$. Recall that

$$\Omega^1(P(D_q)) = \{\sum_k a_k \otimes b_k | \sum_k a_k b_k = 0\} \subset P(D_q) \otimes P(D_q)$$

($da = 1 \otimes a - a \otimes 1$). By Lemma 2 we have a left $P(D_q)$-module basis in $P(D_q) \otimes P(D_q)$ formed by the elements $1 \otimes x^k x^{*l}$, therefore the elements $\{d(x^k x^{*l}),\ k, l > 0\}$ form a left $P(D_q)$-module basis in $\Omega^1(P(D_q))$: From $\sum_{kl} a_{kl} d(x^k x^{*l}) = \sum_{kl} a_{kl} \otimes x^k x^{*l} - \sum a_{kl} x^k x^{*l} \otimes 1 = 0,\ a_{kl} \in P(D_q)$, follows $a_{kl} = 0,\ \forall k, l$.

Now we define the following left module projection $P_1 : \Omega^1(P(D_q)) \longrightarrow \{adx + bdx^* | a, b \in P(D_q)\} \subset \Omega^1(P(D_q))$:

$$\begin{aligned} P_1(dx) &:= dx, \quad P_1(dx^*) := dx^* \\ P_1(d(x^k)) &:= \sum_{i=0}^{k-1} q^i x^{k-1} dx, \quad k > 0 \\ P_1(d(x^{*l})) &:= \sum_{i=0}^{l-1} q^{-i} x^{*l-1} dx^*, \quad l > 0 \\ P_1(d(x^k x^{*l})) &:= \sum_{i=0}^{k-1} q^{i-l} x^{k-1} x^{*l} dx + \sum_{i=0}^{l-1} q^{-i} x^k x^{*l-1} dx^*, \quad k > 0;\ l > 0 \end{aligned}$$

All generators of $J(D_q)$ lie in the kernel of P_1 (for example: $P_1(xdx-q^{-1}(dx)x = P_1(q^{-1}((1+q)xdx - dx^2) = 0)$, thus the left module generated by these elements lies in the kernel of P_1. If we can show that also the right module generated by these elements lies in the kernel of P_1, the whole first degree of $J(P(D_q))$ lies in $kerP_1$. We show this for the generator $xdx - q^{-1}(dx)x = (1+q^{-1})xdx - q^{-1}dx^2$ and leave the remaining cases to the reader.

The Leibniz rule gives

$$((1+q^{-1})xdx - q^{-1}dx^2)x^k x^{*l} = (1+q^{-1})xd(x^{k+1}x^{*l}) - x^2 d(x^k x^{*l}) - q^{-1}d(x^{k+2}x^{*l}).$$

Applying P_1 to this formula one gets

$$\begin{aligned} & P_1(((1+q^{-1})xdx - q^{-1}dx^2)x^k x^{*l}) \\ = \; & (1+q^{-1})\sum_{i=0}^{k} q^{i-l}x^{k+1}x^{*l}dx - (1+q^{-1})\sum_{i=0}^{l-1} q^{-i}x^{k+2}x^{*l-1}dx^* \\ - \; & \sum_{i=0}^{k-1} q^{i-l}x^{k+1}x^{*l}dx - \sum_{i=0}^{l-1} q^{-i}x^{k+2}x^{*l-1}dx^* \\ - \; & q^{-1}\sum_{i=0}^{k+1} q^{i-l}x^{k+1}x^{*l}dx - q^{-1}\sum_{i=0}^{l-1} q^{-i}x^{k+2}x^{*l-1}dx = 0, \quad \forall k,l. \end{aligned}$$

The calculation for the remaining generators of $J(P(D_q))$ is analogous, thus

$$J^1(P(D_q)) \subseteq kerP_1. \tag{54}$$

Let $\pi_{\Omega,\Gamma} : \Omega(P(D_q)) \longrightarrow \Gamma(P(D_q))$ be the quotient map. Because of formula (54) there exists a left $P(D_q)$-module homomorphism $\Lambda_1 : \Gamma^1(P(D_q)) \longrightarrow \Omega^1(P(D_q))$ defined by

$$\Lambda_1 \circ \pi_{\Omega,\Gamma} := P_1.$$

Applying Λ_1 to $adx + bdx^* = 0$ it follows that $adx + bdx^* = 0$ in $\Omega^1(P(D_q))$, which gives $a = 0$ and $b = 0$. The right basis property follows now easily from the left one and the relations defining $\Gamma(P(D_q))$.

(ii) Applying the differential d to the generators of $J(P(D_q))$ one obtains the following relations in $\Gamma^2(P(D_q))$:

$$dxdx = 0; \quad dx^*dx^* = 0; \quad dx^*dx = -qdxdx^*.$$

The procedure of the proof of (i) carries over to the degree two case:

First one proves that the elements $d(x^m x^{*n})d(x^k x^{*l})$ form a left $P(D_q)$-module basis of $\Omega^2(P(D_q))$. Then one defines a left $P(D_q)$-module projection
$P_2 : \Omega^2(P(D_q)) \longrightarrow \{adxdx^*,\ a \in P(D_q)\} \subset \Omega^2(P(D_q))$:

$$\begin{aligned} & P_2(dxdx^*) := dxdx^*, \quad P_2(dx^*dx) := -qdxdx^* \\ & P_2(dx^m dx^k) := 0, \quad P_2(dx^{*n}dx^{*l}) := 0 \\ & P_2(dx^m dx^{*l}) := q^{1-l}\sum_{i=0}^{m-1}\sum_{s=0}^{l-1} q^{i-s}x^{m-1}x^{*l-1}dxdx^*, \quad m,l \geq 1 \\ & P_2(dx^{*n}dx^k) := -q^k\sum_{i=0}^{n-1}\sum_{s=0}^{k-1} q^{s-i}x^{*n-1}x^{k-1}dxdx^*, \quad n,k \geq 1 \\ & P_2(d(x^m x^{*n})d(x^k x^{*l}) := q^{k-l}\sum_{i=0}^{m-1}\sum_{s=0}^{l-1} q^{i-n}q^{1-s}x^{m-1}x^{*n}x^k x^{*l-1}dxdx^* \\ - \; & q^{k-l}\sum_{i=0}^{n-1}\sum_{s=0}^{k-1} q^{-i}q^{s-l}x^m x^{*n-1}x^{k-1}x^{*l}dxdx^*, \quad m,n,k,l \geq 1. \end{aligned}$$

With these definitions one proves that $J^2(P(D_q)) \subseteq ker P_2$.

Thus, there exists a left $P(D_q)$-module homomorphism $\Lambda_2 : \Gamma^2(P(D_q)) \longrightarrow \Omega^2(P(D_q))$ defined by

$$\Lambda_2 \circ \pi_{\Omega,\Gamma} := P_2; \quad (\Lambda_2(adxdx^*) = adxdx^*)$$

Applying Λ_2 to $adxdx^* = 0, \quad a \in P(D_q)$ in $\Gamma(P(D_q))$ it follows that $adxdx^* = 0$ in $\Omega^2(P(D_q))$, which gives $a = 0$. Again, the right basis property is now easily derived.

(iii) Immediate. $\square$

Acknowledgments: We thank P. M. Alberti and P. M. Hajac for helpful discussions and P. Podleś and W. Pusz for valuable remarks.

References

[1] Atiyah, M. F. and I. G. Macdonald: Introduction to commutative algebra, Addison Wesley Publishing Company 1969

[2] Budzynski, R. J. and W. Kondracki: Quantum principal fiber bundles: Topological aspects, Rep. Math. Phys. **37** (1996), 365-385 preprint 517, PAN December 1993, hep-th/9401019

[3] Calow, D. and Matthes, R.: Connections on locally trivial principal fibre bundles, in preparation

[4] Dixmier, J.: Les C^*-algebres et leurs representations, Gauthier-Villars, Paris 1964

[5] Klimek, S. and A. Lesniewski: Quantum Riemann surfaces I. The unit disc, Comm. Math. Phys. **146** (1992), 103-122

[6] Klimek, S. and A. Lesniewski: A two-parameter quantum deformation of the unit disc, J. Funct. Anal. **115** (1993), 1-23

[7] Masuda, T., Nakagami, Y. and J. Watanabe: Noncommutative differential geometry on the quantum two sphere of Podles. I: An algebraic viewpoint, K-theory **5** (1991), 151-175

[8] Pflaum, M. J. and P. Schauenburg: Differential calculi on noncommutative bundles, Preprint- Nr gk-mp-9407/7, München 1994

[9] Podles, P.: Quantum spheres, Lett. Math. Phys. **14** (1987), 193-202

[10] Seibt, P.: Cyclic homology of algebras, World Scientific Singapore New Jersey Hong Kong 1987

[11] Sheu, A. J.-L.: Quantization of the Poisson SU(2) and its Poisson homogeneous space - the 2-sphere, Commun. Math. Phys. **135** (1991), 217-232

[12] Sinel'shchikov, S. and L. Vaksman: On q-analogues of bounded symmetric domains and Dolbeault Complexes, Mathematical Physics, Analysis and Geometry, **1** (1)(1998), 75-100, q-alg/9703005

[13] Zariski, O. and P. Samuel: Commutative algebra, vol. I, 1958, D. van Nostrand, Inc., Princeton, Toronto, London, New York.